# Adaptive Learning and Macroeconomics

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July 10, 2020

**Abstract**

While rational expectations (RE) remains the benchmark paradigm in macroeconomic modeling, bounded rationality, especially in the form of adaptive learning, has become a mainstream alternative. Under the adaptive-learning (AL) approach, economic agents in dynamic, stochastic environments are modeled as adaptive learners forming expectations and making decisions based on forecasting rules that are updated in real time as new data become available. Their decisions are then coordinated each period via the economy's markets and other relevant institutional architecture, resulting in a time-path of economic aggregates. In this way, the AL approach introduces additional dynamics into the model – dynamics that can be used to address myriad macroeconomic issues and concerns, including, for example, empirical fit and the plausibility of specific rational expectations equilibria.

AL can be implemented as *reduced-form learning*, that is, the implementation of learning at the aggregate level; or, alternatively, as discussed in a companion contribution, Evans and McGough (2020a), as *agent-level learning*, which includes pre-aggregation analysis of boundedly rational decision making.

Typically learning agents are assumed to use estimated linear forecast models, and a central formulation of AL is least-squares learning in which agents recursively update their estimated model as new data become available. Key questions include whether AL will converge over time to a specified RE equilibrium (REE), in which cases we say the REE is stable under AL; in this case it is also of interest to examine what type of learning dynamics are observed en route. When multiple REE exist, stability under AL can act as a selection criterion, and global dynamics can involve switching between local basins of attraction. In models with indeterminacy, AL can be used to assess whether agents can learn to coordinate their expectations on sunspots.

The key analytical concepts and tools are the E-stability principle together with the E-stability differential equations, and the theory of stochastic recursive algorithms

(SRA). While in general analysis of SRAs is quite technical, application of the E-stability principle is often straightforward.

In addition to equilibrium analysis in macroeconomic models, AL has many applications. In particular AL has strong implications for the conduct of monetary and fiscal policy, has been used to explain asset price dynamics, has been shown to improve the fit of estimated DSGE models, and has been proven useful in explaining experimental outcomes.

Keywords: Bounded rationality; E-stability; least-squares learning; temporary equilibrium; rational expectations equilibrium; multiple equilibria; macroeconomic policy with adaptive learning.

# Introduction to Adaptive Learning

Macroeconomic models are usually based in recursive, stochastic settings and can be summarized by dynamic systems that include expectational dependencies. In the simplest case of point expectations and representative agents, this could take the form

$$y_t = Q(y_{t-1}, y^e_{t+1}, w_t),$$

where $y_t$ is a vector of endogenous variables at time $t$ (unemployment, inflation, investment, etc.), $y^e_{t+1}$ denotes the expectations formed at time $t$ of the values these variables take at time $t+1$, and $w_t$ is an exogenous vector of random factors at $t$. A more general formulation

$$E^*_t F(y_t, y_{t-1}, y_{t+1}, w_t) = 0 \tag{1}$$

allows for expectations of nonlinear interactions of future variables, where $E^*_t$ denotes the subjective expectation of the representative agent. In many models heterogeneity across agents is also important, as discussed later.

The presence of expectations, which typically arises from the (possibly implicit) assumption that the model's forward-looking agents solve dynamic programing problems, makes macroeconomics inherently different from natural science, and thus necessitates distinct solution concepts and methods.[1] Solving a model of the form (1) requires taking a stand on how agents form expectations. The benchmark "rational expectations" (RE) assumption in macroeconomics is that agents' expectations are formed optimally given their information; said differently, $E^*_t$ is taken to be the mathematical expectations operator conditional on period $t$ information. RE, as it is commonly implemented, requires that agents form their expectations against the objective conditional distribution of the model's variables, including $y_{t+1}$. However, this distribution is endogenous, and in particular depends on the manner in which all agents form expectations. Thus RE, as usually implemented in macroeconomics,

[1] Evans and Honkapohja (2009b) and Woodford (2013) provide recent surveys that emphasize expectations, learning and bounded rationality in macroeconomics.

is an equilibrium object: it is not (in general) rational to have rational expectations unless everyone else has rational expectations.

An alternative to RE is the adaptive learning approach, a perspective on expectation formation that views the standard assumption of RE as demanding too much knowledge of, and coordination by, economic agents: the endogenous distribution characterizing a rational expectations equilibrium (REE) is an unobserved solution to an infinite-dimensional system of equations that, in general, even the modeler can only approximate; furthermore, a mechanism would need to be specified for how agents could coordinate on the REE. Clearly a more realistic model of behavior is required. What form should this take?

One answer to this question is given by the *Cognitive Consistency Principle*: economic agents should be about as smart as economists. While this principle might be implemented in various ways, here the focus is on the adaptive learning (AL) approach, which takes agents as revising their forecasting models and decision rules as new data become available. Usually agents are assumed to act as time-series econometricians, updating their model parameters over time, but the adaptive learning approach also includes, for example, selection among alternative behavioral rules based on recent forecast success. Reflecting the view that RE implicitly requires agents to have an unreasonable degree of knowledge about the structure of the economy, the adaptive learning approach instead typically assumes that agents make forecasts by simply regressing the variables being forecasted on relevant observed exogenous or lagged endogenous variables, and updating the estimated coefficients over time.

There has been a wide range of applications of AL in macroeconomics. Two particularly notable areas in which AL has had major impacts are (i) the implementation of monetary policy and (ii) the empirical fit of macro and macro-finance models.

The key insight of AL for monetary policy is the failure of the RE viewpoint to recognize that expectations can have an autonomous impact on the economy. In the monetary policy context this point is implicit in Orphanides and Williams (2005) and Bullard and Mitra (2002). Orphanides and Williams (2005) showed under least-squares learning expectations could deviate substantially, for example leading to high inflation extended periods, unless the policy rule was sufficiently aggressive in counteracting deviations of inflation from the central bank target. Bullard and Mitra (2002) showed, for the standard NK model, that the specification of the Taylor rule had to be chosen to ensure that the targeted REE is locally stable under learning. In particular, uniqueness of the REE does not ensure it is attainable. Furthermore, Evans and Honkapohja (2003c) show, in the context of optimal policy, that stability under learning must be taken into account in the implementation of policy: there are policy rules consistent with optimal policy that are unstable and hence unattainable.[2] A related issue has recently become topical: the efficacy of interest-rate pegs. The neo-Fisherian view advises adopting a higher interest-rate peg in order to raise inflation to its target. Evans and McGough (2018b) and Evans and McGough (2018a) demonstrate the

[2]Optimal policy under adaptive learning is also studied in Evans and McGough (2007), Molnar and Santoro (2014) and Eusepi, Giannoni, and Preston (2018).

dangers of this policy prescription by establishing that interest-rate pegs lead to economic instability because of divergent expectation dynamics.[3]

The potential for AL to improve empirical macro models became evident in Sargent (1999) and Ireland (2003). A prominent early example is the hyperinflation paper of Marcet and Nicolini (2003). These authors show that in the standard hyperinflation model, adaptive learning agents fits well the recurring episodes of hyperinflation in South America, whereas the same model with rational agents has implications that are at odds with the stylized facts from the data.[4] Milani (2007) uses Bayesian techniques to compare the fit of a standard New Keynesian model with adaptive vs. rational agents. In addition to showing an improved fit by AL, he finds that the AL model does not require the "mechanical sources of persistence (habits, indexation)" needed under RE to match the data.[5] More recently, Eusepi and Preston (2011) show improved fit of AL over RE using a calibrated expectations-driven model of the business cycle.[6] The potential for AL to explain financial data was first emphasized by Timmermann (1993), who used learning dynamics to explain asset-price volatility: see also Timmermann (1994) and Timmermann (1996). Subsequent work includes applications to the forward-premium puzzle, the yield curve and stock-price bubbles: see, respectively, Chakraborty and Evans (2008), Sinha (2016) and Branch and Evans (2011), Lansing (2010). A compelling case for the role of adaptive learning in explaining US asset-pricing facts is provided by Adam, Marcet, and Nicolini (2016).

## *Reduced-form learning versus the agent-level approach*

The implementation of AL in ad hoc macro models, i.e. models that take as primitive the relationships between aggregates, is straightforward in concept: simply replace the conditional expectations operator with a boundedly rational counterpart. Micro-founded dynamic stochastic general equilibrium (DSGE) environments allow for this method of implementation as well, but also invite a more nuanced approach that explicitly models agent behavior. Put more succinctly, in micro-founded models the AL approach can be implemented at the aggregate (reduced-form) level or at the agent level. This chapter focuses on *reduced-form learning* (RFL), i.e on implementation at the aggregate level. Agent-level considerations are left to a companion chapter, Evans and McGough (2020a). However, to motivate RFL, and

[3] Eo and McClung (2020) examine determinacy and stability under learning when temporary interest-rate pegs are part of a regime-switching policy environment.

[4] Hyperinflations in South America were also studied by Sargent, Williams, and Zha (2009) using calibrated and estimated models. In a similar model Evans, Honkapohja, and Marimon (2001) include experimental evidence supporting the model with AL.

[5] This paper is a component of a larger program that includes, for example, Milani (2011). Separately Slobodyan and Wouters (2012a) find that the empirical fit of the Smets and Wouters (2007) model is improved with adaptive learners, including in particular for the mean and volatility of inflation. See Section  for further discussion.

[6] See also Eusepi and Preston (2018a), which emphasizes the interaction between adaptive learning and government debt in explaining inflation.

to draw distinction between it and agent-level implementations, it is helpful to begin with a broader discussion that includes boundedly optimal decision making.

The modern, micro-founded DSGE approach to macroeconomic modeling and analysis includes a common architecture comprised of many agents making decisions in dynamic, stochastic environments, with the per-period determination of aggregates coordinated by markets and possibly other institutional arrangements. This per-period determination, referred to as *temporary equilibrium*, results in a time-indexed collection of equations and inequalities characterizing the model's endogenous aggregates as functions of the exogenous drivers.[7]

The common architecture described above must include assumptions detailing *how* agents make decisions. The benchmark is to adopt the *rational expectations hypothesis* which stipulates that agents forecast optimally given their information and make decisions optimally given their forecasts – in effect, to make their period $t$ decisions agents are assumed to solve a dynamic programming problem, taking into account the stochastic evolution of all the relevant state variables, including the economy's endogenous aggregates. Temporary equilibrium is then obtained by imposing market clearing and aggregating.

Because agents' behavior depends on their expectations of future aggregates, which, themselves, depend on agents' behavior, an additional equilibrium concept is needed to guarantee internal consistency. A *rational expectation equilibrium* (REE) is a distribution over the collections of time-paths of endogenous variables that is consistent with rational decision-making in this sense that if agents condition on the distribution then their attendant decisions will aggregate to comport with the distribution.

Explicit, closed-form computation of an REE is very challenging – indeed, it is not possible under most circumstances. In the discrete-time case, the most common approximation approach involves simplifying the temporary equilibrium restrictions to obtain a system of non-linear, expectational difference equations – the reduced-form system – and then linearizing these equations around a non-stochastic steady state, thus yielding a system of linear, expectational difference equations – the linearized reduced-form system – and then finally using well-known techniques to solve this linear RE-model.

The adaptive learning literature questions the wisdom of adopting the rational expectations hypothesis, which itself is tantamount to the *assumption* that the economy is always in an REE. Instead, the AL literature prefers the view that an REE is a possible outcome that might, or might not, *emerge* as a result of more realistically modeled agent-level decision making.

A natural implementation of adaptive learning thus begins by discarding the rational expectations hypothesis, and instead developing explicit models of boundedly rationality, including alternate specifications of forecasting models, planning horizons and decision rules. The implied decisions are then coordinated and aggregated in temporary equilibrium, thus

[7] The notion of temporary equilibrium was introduced by Hicks (1946).

resulting in a dynamic system that can at least be simulated if not analytically assessed. This is the agent-level approach to adaptive learning, and includes, for example, Euler-equation learning (Evans and Honkapohja (2006)), long-horizon learning (Preston (2005)), and the shadow-price approach developed by Evans and McGough (2018c). Details of agent-level learning are explored in Evans and McGough (2020a).

An alternative and simpler approach is reduced-form learning. Under this implementation, the reduced-form system of expectational difference equations, derived under rational expectations, is taken as given. The modeler then simply replaces the conditional expectations operators in this reduced form system with a boundedly rational counterpart, and proceeds with analysis. In fact, this replacement is often done after the reduced-form system is linearized.

RFL can be viewed as a reasonable and effective short cut that can greatly simplify the analysis of a given DSGE model; and, if needed, it can often be justified via an agent-level approach; and it is particularly convenient when conducting empirical work. The agent-level approach is more appropriate for policy analysis, particularly in models with complex agent-level behaviors that are expected to impinge on policy outcomes.

### *The appeal of adaptive learning*

By anchoring to behavioral assumptions that are more plausible than those required by the rational expectations hypothesis, the AL approach has considerable intellectual appeal. Additional attractive features include:

1. AL provides a test of the plausibility of an REE. Because the economy is self-referential in the sense that the evolution of the economy depends on the expectations of agents and vice versa, an REE is most naturally viewed as a possible emergent outcome of this evolution. AL provides a natural mechanism through which an REE might emerge, and thereby provides a "plausibility test" of the REE.

2. Many macro models have multiple REE. With multiple REE it is common for one or more of the REE to be (locally) stable under AL while stability fails to hold for other REE. In these circumstances AL acts as a selection criterion.

3. Under AL there are learning dynamics, at least in the transition to RE. Furthermore, natural relaxations of adaptive learning rules can lead to persistence in these dynamics, which can help fit empirical regularities.

4. In most standard macro models, RE implies homogenous expectations about key economic aggregates. This is counterfactual: survey evidence always shows substantial heterogeneity. AL can be easily modified in simple and natural ways to allow for heterogeneous expectations.

5. The temporary equilibrium AL approach is readily implemented computationally, and is often more tractable than RE, especially in nonlinear environments.

### *Alternatives to adaptive learning*

While the focus of this chapter is on the AL approach summarized above, it is important to note that there are related approaches to expectation formation that are distinct but complementary. The eductive approach, introduced by Guesnerie (1992), considers conditions under which fully rational agents, with common knowledge both of the economic structure and of the rationality of other agents, would be able through mental reasoning to coordinate on an REE. The conditions for coordination, which can be quite strict, are related to the iterative formulation of expectational stability used by Evans (1985); see the discussion in Evans and Guesnerie (1993). Applications of the eductive stability or "strong rationality" approach can be found in Guesnerie (2005). For an exploration of the complementary insights of eductive and adaptive approaches in the context of the Real Business Cycle model, see Evans, Guesnerie, and McGough (2019).[8]

A related model of expectation formation is the "k-level" approach of Nagel (1995).[9] As in the eductive approach, agents need structural knowledge of the economy and engage in higher level reasoning, but different agents may choose to use different levels of reasoning. Evans, Gibbs, and McGough (2019) provides a model that combines adaptive and k-level reasoning in which agents may switch their choice of k-level based on past performance. Finally, as discussed in the Section titled *Behavioral Rules*, there is a range of behavioral approaches in which agents make decisions based on forecasts made using simple rules of thumb: see the recent survey Hommes (2020). To the extent that agents choose rules based on past performance, this is consistent with the spirit of the AL approach.

### *Chapter organization*

This overview of adaptive learning in macroeconomics begins with a description of the key equilibrium and stability tools, and related techniques, within a particularly simple framework. Next these tools are extended to a general multivariate linear setting, using the reduced-form short-cut, and then applied to the analysis of monetary policy in a New Keynesian setting. Models in which there can be multiple REE are then discussed. AL is shown to be useful as a selection criterion for assessing whether there exist stable "sunspot equilibria" in which economic fluctuations are driven by self-fulfilling expectations. Finally, the Section titled *Other Applications and Extensions* concludes with a survey of applied research

[8] For additional papers on these topics, see also Ellison and Pearlman (2011), Gaballo (2013), Gaballo (2014) and Bao and Duffy (2016).

[9] Closely related approaches are developed in Evans and Ramey (1992), Garcia-Schmidt and Woodford (2019) and Fahri and Werning (2019).

that uses AL to study policy and related issues.

## Adaptive Learning in the Cobweb Model

Developing the AL approach requires some initial investment in tools. The key AL concepts can be illustrated using the linear "cobweb" model examined by Muth (1961) in his seminal formulation of RE. The cobweb model considers an isolated market for a perishable good. Demand for the good is linear, taking the form

$$D_t = A - Bp_t + v_{1t}, \text{ where } A, B > 0,$$

and there is a continuum of firms $i \in [0,1]$ that supply output in a competitive market. Production is subject to a one period delay, so that the supply of firm $i$ depends on the expected price $p_t^e(i)$, and the cost structure is such that supply depends linearly on expected price. Here $v_{1t}$ is a zero mean *iid* shock and, with suitable assumptions, $p_t$ will be such that $D_t$ is always positive.

Suppose that the cost of producing planned quantity $S_t^*(i)$, for each firm $i$, is given by $G_1 S_t^*(i) + \frac{1}{2} G_2 \left(S_t^*(i)\right)^2 - H' w_{t-1} S_t^*(i)$, and that there is also a zero mean *iid* shock $v_{2t}$ that affects the actual quantity supplied, i.e. $S_t(i) = S_t^*(i) + v_{2t}$. Assuming firms are risk neutral and at time $t-1$ choose $S_t^*(i)$ to maximize expected profits, the quantity $S_t(i)$ supplied by firm $i$ is

$$S_t(i) = Cp_t^e(i) + K + F' w_{t-1} + v_{2t}$$

where $C = G_2^{-1} > 0$, $K = -G_2^{-1} G_1$ and $F = G_2^{-1} H$. The exogenous observable vector $w_t$ represents factors that affect marginal costs. It is assumed to follow a stationary stochastic process and is assumed to have zero mean and finite second moments. For example, $w_t$ could follow a stationary VAR(1) process.

Let $p_t^e(i)$ denote the expectation of firm $i$ of $p_t$, the market-clearing price at $t$. Assume that $p_t^e(i)$ is formed at the end of period $t-1$. This expectation can be interpreted as the mean of $p_t$, conditional on information available at $t-1$, computed using the subjective probability distribution of firm $i$. It is convenient to write $E_{t-1}^* p_t(i)$ for $p_t^e(i)$, or, in the homogeneous expectations case, $E_{t-1}^* p_t = p_t^e$. In general one can allow for heterogeneous expectations across firms, with aggregate supply is given by $S_t = \int_0^1 S_t(i) di$. Assuming homogenous expectations case, period $t$ price is determined by market clearing, i.e. $D_t = S_t$, which gives the equation

$$p_t = \mu + \alpha p_t^e + \delta' w_{t-1} + \eta_t, \tag{2}$$

where $\mu = B^{-1}(A - K)$, $\alpha = -B^{-1}C$, $\delta = -B^{-1}F$ and where $\eta_t = B^{-1}(v_{1t} - v_{2t})$ is *iid* exogenous with mean zero. Note that $\alpha < 0$ assuming demand and supply relations have the usual slopes. Equation (2) is the temporary equilibrium (TE) equation that determines market clearing price, given expectations and the exogenous shocks.

Under RE, replacing $p_t^e = E_{t-1}^* p_t$ by $E_{t-1}p_t$, the true conditional distribution of $p_t$, it is easily verified that there is a unique REE in which

$$E_{t-1}p_t = \bar{a} + \bar{b}' w_{t-1}, \text{ where } \bar{a} = (1-\alpha)^{-1}\mu \text{ and } \bar{b} = (1-\alpha)^{-1}\delta, \text{ and} \tag{3}$$

$$p_t = \bar{a} + \bar{b}' w_{t-1} + \eta_t.$$

It can be seen, via the TE map (2), the sense in which RE must be viewed as an equilibrium concept. Indeed, suppose all agents $i \in [0,1]$ except for agent $j$ had naive expectations, i.e. $p_t^e(i) = p_{t-1}$ for all $i \neq j$. Then $p_t^e = p_{t-1}$ so that

$$p_t = \mu + \alpha p_{t-1} + \delta' w_{t-1} + \eta_t,$$

from which it follows that the rational forecast for agent $j$ is

$$p_t^e(j) = \mu + \alpha p_{t-1} + \delta' w_{t-1},$$

which is different from RE.

## *Behavioral rules*

The cobweb example illustrates the flexibility of the temporary equilibrium approach, in that one can examine the implications of agents using forecasts that reflect plausible behavioral rules or rules of thumb. Economic "learning to forecast" experiments show that subjects often appear to use simple forecast rules. Examples include

$$\begin{aligned} \text{Naive} &: \; p_t^e = p_{t-1} \\ \text{Trend-chasing} &: \; p_t^e = p_{t-1} + \vartheta\left(p_{t-1} - p_{t-2}\right), \text{ where } 0 < \vartheta < 1 \\ \text{Adaptive expectations} &: \; p_t^e = p_{t-1}^e + \lambda\left(p_{t-1} - p_{t-1}^e\right), \text{ where } 0 < \lambda < 1 \\ \text{Mean forecasts} &: \; p_t^e = N^{-1}\sum\nolimits_{j=1}^{N} p_{t-j}, \text{ for integer } N > 1. \end{aligned}$$

For the apparent prevalence of these types of rules used by subjects in experiments see Hommes (2011), Hommes (2013) and . In these experiments, serially correlated exogenous observables are typically not included. It should be noted that in some circumstances such behavioral rules can be fully optimal. For example, adaptive expectations (AE) is known to provide the optimal forecast, for appropriate $\lambda$, if $p_t$ follows an IMA(1,1) process, i.e. if the first-difference of $p_t$ is a first-order moving average process.

For any of the above forecast rules, one can solve for the implied TE path by inserting the expectation rule into the TE equation (2). Indeed, it is also straightforward computationally to obtain the implied TE path that arises when there are heterogeneous expectations and fixed proportions of agents that use, for example, each of the above four forecast rules.

From the AL viewpoint this falls short of the above bounded rationality guidelines since each agent is using a fixed rule without trying to choose a well-performing forecast rule.

However, some natural extensions of the behavioral approach clearly do satisfy the principle outlined above. If agents used AE but instead of using a fixed $\lambda$ they attempted to use the best-performing value of $\lambda$, this would be an AL approach. Similarly what is called "mean forecasts" can be viewed as providing an optimal estimate of the population mean of the $p_t$. Finally if one allows agents to *choose* between the above four rules, and if their choices reflect relative forecast performance, then this more sophisticated version of forecasting also is in line with the the AL guiding principles. Some evidence of shifting proportions over time in the choice of rules can be found in Hommes (2013).

## *Least-squares learning*

The lead implementation of AL for the exposition here will be econometric learning, and more specifically least-squares (LS) learning. This implementation obeys the cognitive consistency principle: when economists need to forecast they usually proceed statistically, and the benchmark procedure is to use LS. Least-squares is a central forecasting procedure because of its optimality properties, e.g. the minimum mean-square-error linear forecast of a variable $y$ given observed variables $x$ is provided by the least-squares projection of $y$ onto $x$. This linear projection can be estimated easily using the standard ordinary least squares regression of $y$ onto $x$ using the available data.

For the cobweb model under consideration here, agents (here, the firms) are assumed to have a Perceived Law of Motion (PLM)

$$p_t = a + b' w_{t-1} + \eta_t \tag{4}$$

where $a, b$ are unknown and $\eta_t$ is a perceived white noise (exogenous *iid* zero mean) unobserved shock. Thus agents are assumed to understand that specified observable, exogenous variables $w_{t-1}$ impact equilibrium price in a linear way, but that agents do not know the RE values $\bar{a}, \bar{b}$. Under LS learning, the agents will estimate $a, b$ using time-series data on $p_t, w_{t-1}$. Suppose that agents use their estimates to make forecasts and update these estimates over time as more data become available. Will their estimates converge over time to the REE values? To answer this one uses the TE framework with expectations evolving in accordance with LS learning.

Start the system at time $t-1$ with initial data $\{p_s, w_{s-1}\}_{s=1}^{t-1}$ and assume that in $t-1$ agents first use these data to regress $p$ on lagged $w$ yielding estimates $(a_{t-1}, b_{t-1})$ of the unknown $(a, b)$, according to

$$\begin{pmatrix} a_{t-1} \\ b_{t-1} \end{pmatrix} = \left( \sum_{s=1}^{t-1} z_{s-1} z_{s-1}' \right)^{-1} \left( \sum_{s=1}^{t-1} z_{s-1} p_s \right), \text{ where } z_s' = \begin{pmatrix} 1 & w_s' \end{pmatrix}. \tag{5}$$

The exogenous vector $w_{t-1}$ is then realized, and time $t-1$ expectations of $p_t$ are formed according to

$$p_t^e = a_{t-1} + b_{t-1}' w_{t-1}. \tag{6}$$

Move next to period $t$. Given expectations (6) and the white noise shock $\eta_t$, the temporary equilibrium price $p_t$ is determined by (2). The data set then incorporates $(p_t, w_{t-1})$, the LS coefficients estimates $(a_{t-1}, b_{t-1})$ are updated to $(a_t, b_t)$, given by

$$\begin{pmatrix} a_t \\ b_t \end{pmatrix} = \left( \sum_{s=1}^{t} z_{s-1} z'_{s-1} \right)^{-1} \left( \sum_{s=1}^{t} z_{s-1} p_s \right), \tag{7}$$

and the process continues. This recursion fully defines a path for $a_t, b_t, p_t$ over time. The question of interest is whether

$$(a_t, b_t) \rightarrow (\bar{a}, \bar{b}) \text{ as } t \rightarrow \infty.$$

If so the REE is said to be stable under LS learning. The answer is given by the following Theorem, due to Bray and Savin (1986) [10]and Marcet and Sargent (1989b).

**Theorem 1** *Consider the dynamic system given by (2) and (6) with LS updating of* $(a_t, b_t)$ *according to (5). If* $\alpha < 1$ *then* $(a_t, b_t) \rightarrow (\bar{a}, \bar{b})$ *as* $t \rightarrow \infty$ *with probability 1. If* $\alpha > 1$ *then* $(a_t, b_t)$ *then convergence occurs with probability 0.*

Note that if supply and demand have their usual slopes then $\alpha < 0$ so that convergence to RE occurs with probability one. The case $\alpha < 0$ is often called the *negative expectational feedback* case. Some simple macro models take the form (2) but have *positive expectational feedback.* Convergence of LS learning to the REE still obtains if $0 < \alpha < 1$.

As an example with positive feedback, a simple ad hoc macro model combines an expectations augmented Phillips curve with a quantity theory aggregate demand equation and a money supply feedback rule is given by:

$$\begin{aligned} y_t &= \bar{y} + \kappa \left( p_t - p_t^e \right) + \zeta_t, \;\; \kappa > 0 \\ m_t + v_t &= p_t + y_t \\ m_t &= \bar{m} + u_t + \psi w_{t-1}, \end{aligned}$$

where $w_t$ is an exogenous observable vector and $\zeta_t, v_t$ are exogenous white noise. When solved for $p_t$ this takes the form (2) with $\alpha = (1+\kappa)^{-1} \kappa$, giving us the stable positive feedback case.

The positive result in Theorem 1 was proved from first principles by Bray and Savin (1986). Marcet and Sargent (1989b) provided a more general framework, based on stochastic approximation theorems, which also delivers the negative result. The latter theorems are discussed in detail in Evans and Honkapohja (2001).

[10] Bray and Savin point out that their implementation of least-squares learning can be interpreted as Bayesian estimation with diffuse priors. Bullard and Suda (2016) demonstrate the nuances of Bayesian learning in a forward-looking model.

## ***E-stability***

Formal proofs of the results in the Section titled *Least-squares Learning* are based techniques that require some sophisticated technical machinery; however, the stability condition can generally be obtained quickly using the *expectational stability* or *E-stability* principle. The E-stability approach is described in Evans and Honkapohja (2001) and has been applied in a wide range of economic models.

For the model (2) this works as follows. Start with PLM (4) and suppose $(a, b)$ are fixed at some (possibly non-REE) value. For this PLM the expectation is given by $E^*_{t-1}p_t = a + b'w_{t-1}$, which, when inserted into (2), leads to the associated data-generating process, or *Actual Law of Motion* (ALM),

$$p_t = \mu + \alpha(a + b'w_{t-1}) + \delta' w_{t-1} + \eta_t. \tag{8}$$

This gives the mapping $T$: PLM $\to$ ALM

$$T(a, b) = (\mu + \alpha a, \delta + \alpha b). \tag{9}$$

Note that the REE $\bar{a}, \bar{b}$ is a fixed point of $T$. E-stability is defined by the ordinary differential equation (ODE)

$$\frac{d}{d\tau}(a, b) = T(a, b) - (a, b), \tag{10}$$

where $\tau$ is notional time. The REE $\bar{a}, \bar{b}$ is *E-stable* if it is a stable fixed point of this ODE. For the model at hand $T$ is linear and the REE is E-stable when $\alpha < 1$. Note that this E-stability condition is precisely the condition given in the above theorem.

The intuition of E-stability is that under LS learning the parameters $a_t, b_t$ are slowly adjusted, on average, in the direction of the corresponding ALM parameters. The *E-stability principle* is that E-stability quite generally governs stability of an REE under LS and closely related adaptive learning rules. The E-stability technique can be used in multivariate linear models, nonlinear models, and if there are multiple equilibria.

## ***Recursive LS and stochastic approximation***

The formal link between E-stability and stability under LS learning starts with the recursive formulation of LS. Letting $\phi'_t = (a_t, b'_t)$ denote the vector of parameter estimates and $z'_{t-1} = (1, w'_{t-1})$ the vector of regressors, the temporary equilibrium equation (8), using time $t-1$ parameter estimates to make forecasts $E^*_{t-1}p_t = a_{t-1} + b'_{t-1}w_{t-1}$, can be written

$$p_t = T(\phi_{t-1})' z_{t-1} + \eta_t, \tag{11}$$

where $T$ is given by (9). Under RLS updating the LS estimates (7) can be written as

$$\begin{aligned} \phi_t &= \phi_{t-1} + t^{-1}R_t^{-1}z_{t-1}(p_t - \phi'_{t-1}z_{t-1}) & (12) \\ R_t &= R_{t-1} + t^{-1}(z_{t-1}z'_{t-1} - R_{t-1}), & (13) \end{aligned}$$

which is known as the recursive least-squares (RLS) system. Here $R_t$ is an estimate of the second moment matrix of the regressors. The RLS system constitutes a *stochastic recursive algorithm* (SRA), as shown by Marcet and Sargent (1989b).[11] There are general methods for analyzing the dynamics of SRAs; see Ljung (1977), Marcet and Sargent (1989b), Benveniste, Metivier, and Priouret (1990) and Evans and Honkapohja (2001). These methods, often called the stochastic approximation techniques, approximate SRAs by an associated ODE. In the context of (12)-(13) the ODE takes the form

$$\begin{aligned} d\phi/d\tau &= R^{-1}M(T(\phi)-\phi), && (14)\\ dR/d\tau &= M-R, && (15)\end{aligned}$$

where $M$ is the unconditional second-moment matrix of the exogenous regressors $z_t$. Here $\tau$ is often interpreted as notional time, but as noted below, $\tau$ can be linked to real time $t$. The steps for obtaining the ODE approximation (14)-(15) are given in the references just cited. For a compact summary see Evans and Honkapohja (2009b).

Inspecting the differential equation (15), it is seen that $\lim_{\tau\to\infty} R(\tau) = M$. Hence $R^{-1}M \to I$ and local stability under (14)-(15) of the fixed point $\bar{\phi}' = \left(\bar{a}, \bar{b}\right)$ is determined by local stability under the "small" ODE

$$d\phi/d\tau = T(\phi) - \phi, \qquad (16)$$

which is identical to the E-stability equation (10). Note that the unique REE for model (2) is the unique fixed point $\bar{\phi} = T(\bar{\phi})$ of the system (10). Stochastic approximation results can be used to show that local convergence with probability one obtains if $\alpha < 1$ whereas there is convergence with probability zero if $\alpha > 1$.[12] This confirms for the REE in the cobweb model that E-stability governs stability under LS learning.

A variation of LS learning that can be sometimes be both useful and simpler, is *generalized stochastic gradient learning.*[13] In this case $R_t$ is replaced in (14) by some fixed matrix and thus there is no additional updating equation (15). If the regressors are exogenous, then it would natural to replace $R_t$ by $M$, the second-moment matrix of the regressors, if is known. In that case the RLS updating equation is just $\phi_t = \phi_{t-1} + t^{-1}M^{-1}z_{t-1}(p_t - \phi'_{t-1}z_{t-1})$ and the associated ODE reduces to (16), i.e. the E-stability equation itself.

[11] SRAs take the general form $\theta_t = \theta_{t-1} + \gamma_t Q(t, \theta_{t-1}, X_t)$, where $\theta_t$ is a vector of parameter estimates, $X_t$ is the state vector, and $\gamma_t$ is a deterministic sequence of "gains." To put (12)-(13) in this form one substitutes $R_t \equiv \tilde{R}_{t-1}$ and sets $\theta'_t = \left(\phi_t, \text{vec}\left(\tilde{R}_t\right)\right)$. Here vec stacks the columns of a matrix into vector form. Substituting for $p_t$ from (11) the SRA form is obtained for suitable $X_t$ and $\gamma_t = t^{-1}$.

[12] In general, various technical assumptions need to be satisfied for application of the stochastic approximation theorems, and these are satisfied for the cobweb model.

[13] See Evans, Honkapohja, and Williams (2010) for an anaysis of generalised stochastic gradient learning.

### *The E-stability principle*

The general definition of E-stability in economic models follows the same steps used in the cobweb model. Given an economic model, consider first an REE, identified as a stochastic process with parameter vector $\bar{\phi}$. Under adaptive learning (typically implemented by LS or a close variant), $\phi$ is estimated by economic agents and the estimates $\phi_t$, which are updated over time, are used to form expectations and make decisions. An REE $\bar{\phi}$ is said to be E-stable if it is locally asymptotically stable under the differential equation (16).[14] To assess E-stability one simply determines the mapping from the PLM parameters $\phi$ to the ALM parameters $T(\phi)$ that describes the actual stochastic process when agents form expectations using $\phi$. E-stability is then determined by checking whether the eigenvalues of $DT(\bar{\phi}) - I$ have negative real parts (or equivalently that eigenvalues of $DT(\bar{\phi})$ having real parts less than one). This, of course, provides the conditions for local stability of $\bar{\phi}$ under the E-stability ODE, and corresponds to local stability under LS learning. In some cases stability of $\bar{\phi}$ under the E-stability ODE is global, in which case one says that $\bar{\phi}$ is globally E-stable for the PLM being considered.

E-stability, which in many cases is a straightforward calculation, quite generally provides the correct condition for stability under LS learning. In many cases stochastic approximation results can be used to demonstrate that stability of an REE under LS learning is governed by E-stability. Even in cases in which for technical reasons the stochastic approximation results cannot be applied, numerical simulations indicate its validity.

A technical detail that can be important in practice arises especially when there are multiple REE associated with a given PLM so that E-stability of an REE is local but not global. In this case the claim that under LS learning E-stability implies convergence with probability one usually requires that the updating rule be augmented by a "projection facility" that prevents estimates from leaving a suitably defined compact set. Weaker convergence results (convergence probabilities near one or positive probability of convergence) can dispense with the projection facility.

It is important to note that E-stability depends on the PLM, i.e. on the specification of the forecasting model used by agents. If there is more than one natural representation of the REE, i.e. more than one PLM consistent with the REE, then in principle the E-stability conditions can depend on the PLM used by agents. For example, if agents overparameterize the dynamics of an REE this can lead to stricter E-stability conditions.[15]

Importantly, the E-stability Principle extends to cases in which agents estimate misspecified models, arising from PLMs that do not nest an REE because they omit a relevant variable, underparameterize the dynamics or misspecify the functional form. Because in practice econometric forecasters recognize that the forecast models they use are, at least to

[14]This definition of E-stability was introduced in Evans (1989) and Evans and Honkapohja (1992). Marcet and Sargent (1989b) emphasized the importance of SRAs to study LS learning.

[15]For these casess one can distinguish between weak and strong E-stability conditions, where the latter refers to E-stability with repsect to the overparameterized specification. See Evans and Honkapohja (2001).

some degree, misspecified, in which the relevant solution concept is a restricted perceptions equilibrium (RPE).[16]

### *Restricted perceptions equilibria*

If the PLM of the agents is linear but misspecified, the E-stability principle extends in a natural way to assess the stability under LS learning of an RPE. The RPE itself can be calculated using a set of orthogonality conditions.

Continuing with the cobweb model (2), suppose $w_t$ is a $2 \times 1$ covariance stationary exogenous vector $w_t' = (w_{1t}, w_{2t})$, where for convenience $Ew_t = 0$, so that

$$p_t = \mu + \alpha E^*_{t-1} p_t + \gamma_1 w_{1,t-1} + \gamma_2 w_{2,t-1} + \eta_t. \tag{17}$$

To capture misspecification, assume agents condition their forecasts only on $w_1$, so that their PLM is given by

$$\text{PLM: } p_t = a + c w_{1,t-1} + \varepsilon_t, \tag{18}$$

where the perceived disturbance $\varepsilon_t$ is treated by agents as unpredictable white noise. Then $E^*_{t-1} p_t = a + c w_{1,t-1}$ and the corresponding ALM is

$$\text{ALM: } p_t = \mu + \alpha a + (\gamma_1 + \alpha c)\, w_{1,t-1} + \gamma_2 w_{2,t-1} + \eta_t. \tag{19}$$

Because the PLM is misspecified the ALM law of motion does not lie in the space of PLMs considered. The RPE coefficients $(a, c)$ that minimize the mean square error (MSE), and the associated E-stability conditions, are obtained from a T-map based on the *projected ALM,*

$$\text{Projected ALM: } p_t = T_a + T_c w_{1,t-1} + \varepsilon_t,$$

which projects the ALM process for $p_t$ (19) onto the variables $(1, w_{1,t-1})$.

The coefficients $(T_a, T_c)$ are given by the least-squares *orthogonality conditions* that the forecast error $p_t - T_a - T_c w_{1,t-1}$ must be uncorrelated with both regressors $(1, w_{1,t-1})$. This leads to the conditions

$$\begin{aligned} E\left(p_t - T^a - T^c w_{1,t-1}\right) &= 0 \\ E\left((p_t - T^a - T^c w_{1,t-1}) w_{1,t-1}\right) &= 0, \end{aligned}$$

where $p_t$ is given by (19). Using $Ew_{1,t} = Ew_{2,t} = E\eta_t = 0$ the two conditions are given by the *projected T-map*

$$T^a = \mu + \alpha a \text{ and } T^c = \gamma_1 + \frac{\omega_{12}}{\omega_{11}} \gamma_2 + \alpha c,$$

where $\omega_{11} = \text{var}\,(w_{1t})$ and $\omega_{12} = \text{cov}\,(w_{1t}, w_{2t})$.

[16] For a general discussion of RPEs see Branch (2006).

The fixed point of the map $T(a,c) = (T_a, T_c)$ is given by

$$\bar{a} = (1-\alpha)^{-1}\mu \text{ and } \bar{c} = (1-\alpha)^{-1}\left(\gamma_1 + \omega_{11}^{-1}\omega_{12}\gamma_2\right).$$

The corresponding E-stability ODE is

$$\frac{d}{d\tau}\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \mu + \alpha a \\ \gamma_1 + \frac{\omega_{12}}{\omega_{11}}\gamma_2 + \alpha c \end{pmatrix} - \begin{pmatrix} a \\ c \end{pmatrix},$$

and it is easily verified that the E-stability condition is once again just $\alpha < 1$. The forecasts $E_{t-1}^{*}p_t = \bar{a} + \bar{c}w_{1,t-1}$ define the RPE expectations, and substituting this expression into (17) gives the RPE solution

$$p_t = \bar{a} + \bar{c}w_{1,t-1} + \gamma_2 w_{2,t-1} + \eta_t.$$

As a second example of an RPE suppose that the cobweb model is nonlinear. More specifically, suppose that demand is given by $D(p_t, v_t)$ where $D_p < 0$, i.e. the demand curve slopes downward, and $v_t$ is an *iid* shock independent of all supply shocks; and assume that supply for each firm is given by $S\left(E_{t-1}^{*}p_t, w_{t-1}\right)$, where the supply curve depend positively on $E_{t-1}^{*}p_t$ and $w_{t-1}$ is a vector of covariance stationary VAR(1) cost shocks. Assume homogeneous expectations across firms. The market equilibrium condition that demand equals total firm supply implies

$$p_t = F\left(E_{t-1}^{*}p_t, w_{t-1}, v_t\right)$$

for a suitable nonlinear $F$. In this example assume that when making forecasts firms do make use of the entire vector $w_{t-1}$, and also that they do so using an estimated linear PLM $p_t = a + b'w_{t-1}$. Under AL their estimates are updated over time using (recursive) LS. For this PLM, the corresponding ALM is given by

$$p_t = F\left(a + b'w_{t-1}, w_{t-1}, v_t\right) \equiv \hat{F}\left(a, b, w_{t-1}, v_t\right).$$

Again, the RPE and its stability properties under learning are obtained using the projected PLM and the E-stability principle. The projected T-map is given by $T(a,b) = \left(T^a, T^b\right)(a,b)$, where

$$T^a(a,b) = E_{w,v}\hat{F}(a,b,w,v) \text{ and } T^b(a,b) = \Sigma_w^{-1}E_{w,v}\left(w\hat{F}(a,b,w,v)\right)$$

where $\Sigma_w$ is the covariance matrix of the supply shocks $w$ and $E_{w,v}$ is the expectation taken over the stationary distributions of $w, v$. The fixed point $\left(\bar{a}, \bar{b}\right)$ of $T$ identifies the RPE associated with the linear forecast model, and it is E-stable if it is locally asymptotically stable under the ODE (10). Evans and McGough (2020b) show that for normally sloped supply and demand curves then, at least for supply shocks with sufficiently small bounded support, there is a unique RPE and it is stable under adaptive learning.

There are many examples and applications of RPE in the literature, e.g. see Marcet and Sargent (1989a), Evans and Honkapohja (2001), Ch. 3 and 11, Evans and Ramey (2006), Branch and Evans (2006a), Adam (2007), Guse (2008), Slobodyan and Wouters (2012a) and Hommes and Zhu (2014). In specific settings RPE are often given specific names, e.g. Evans and McGough (2020c) show existence of "near-rational sunspot equilibria" in nonlinear models when agents have forecasts that depend linearly on continuously-measured extraneous "sunspot" variables.

## *Constant-gain learning*

Under least-squares learning, all past data points count equally, and so when forecasts are updated each period the most recent data point has weight $t^{-1}$. This is reflected in the $t^{-1}$ term in the RLS equations (12)-(13). An alternative that is natural if there is concern that structural change may be occurring over time, in some unmodelled way, is to weight recent data points more heavily than past data points, e.g. to downweight past data points geometrically. This is accomplished by replacing $t^{-1}$ by $\gamma$ for fixed $0 < \gamma < 1$, where $\gamma$ represents the weight on the most recent data point; this in effect gives past data points from time $t - i$ weight $(1 - \gamma)^i$. More generally the term $t^{-1}$ can be replaced by a sequence $\gamma_t > 0$, called the gain sequence, with the main cases being the standard decreasing gain sequence $\gamma_t = t^{-1}$ and the constant gain sequence $\gamma_t = \gamma$.

Typical constant gains for quarterly data in macro models with one-period ahead forecasts are $\gamma \in [0.01, 0.05]$, while for models in which agents make long-horizon forecasts the gains used are smaller. See, for example, Branch and Evans (2006b) and Eusepi and Preston (2011). From the agents' point of view the optimal gain to use depends on the extent of the actual or perceived structural change: a large $\gamma$ will more quickly track resulting changes in optimal forecast parameters, while a small gain is more effective at filtering out noise.

In practice, quantitative and estimated empirical models with adaptive learning typically assume a constant gain. This appears to fit the data well and provides a continuing role for adaptive learning. An advantage of this approach is that one can view the economy as a stationary stochastic process, rather than one exhibiting transitional dynamics. Under constant gain learning, agents estimates are always being revised and hence there is "perpetual learning." Even if the PLM nests the REE, instead of convergence to the RE parameters, estimates will converge to a stochastic process. For small constant gains $\gamma$, the estimates will typically converge to a stochastic process centered at the RE parameters when the RE is E-stable.

Analytical results are available for the case of small constant gain. These results are most easily stated using a modified version of the SRA given by[17]

$$\theta_t = \theta_{t-1} + \gamma Q(t, \theta_{t-1}, X_t).$$

[17] See also the footnote in the Section titled *Recursive LS and stochastic approximation.*

Here for $\theta_t' = \left(\phi_t, \text{vec}\left(\tilde{R}_t\right)\right)$, where $\tilde{R}_{t-1} \equiv R_t$, the associated ODE takes the form

$$d\theta/d\tau = h\left(\theta\left(\tau\right)\right), \text{ where } h_\phi(\phi) = \tilde{R}^{-1}M\left(T\left(\phi\right) - \phi\right) \text{ and } h_{\tilde{R}}(\tilde{R}) = M - \tilde{R}. \qquad (20)$$

The ODE $d\theta/d\tau = h\left(\theta\left(\tau\right)\right)$ give the *mean dynamics* of the discrete-time SRA.

Informally, making the identification $\tau = \gamma t$,[18] it can be shown that, starting from given initial parameter estimates $\theta_0$ near the RE value $\bar{\theta}$, for constant gains $\gamma$ sufficiently small the unconditional expected value of $\theta_t$ can be approximated, over a given time range $0 \leq \tau \leq \mathcal{L}$, by

$$E\theta_t \approx \tilde{\theta}\left(\gamma t, \theta_0\right),$$

where $\tilde{\theta}\left(\tau, a\right)$ is the solution to $d\theta/d\tau = h\left(\theta\left(\tau\right)\right)$ over the same time range. Furthermore, under additional technical assumptions that ensure that a well-defined stationary distribution is reached for large $t$, it can be shown that for $\gamma$ sufficiently small and $\gamma t$ sufficiently large $\theta_t \overset{a}{\sim} N(\bar{\theta}, \gamma C)$ for a suitable matrix $C$. Here $\overset{a}{\sim}$ means "approximately distributed as." See Evans and Honkapohja (2001) for the method for computing $C$.

The mean dynamics given by (20) can be useful for showing the global dynamics of the PLM coefficient estimates, from a variety of initial expectations, as well as for verifying local stability under AL. However, as emphasized in Sargent (1999), even when starting near the REE, particular sequences of random shocks can lead the economy to follow "escape" paths leading to big deviations from the REE steady state for extended periods of time. This possibility of recurrent large deviations, known as "escape dynamics," was documented and studied in Cho, Williams, and Sargent (2002). Using the theory of large deviations, Williams (2019) provides tools for characterizing the frequency and direction of escape dynamics. Applications in which escape dynamics play a prominent role include Kasa (2004), McGough (2006), Cho and Kasa (2008), Branch and Evans (2011) and Branch and Evans (2017).

A motivation for the use of constant-gain learning is the presence of structural change: in non-stationary environments agents have incentive to view apparent outlier data as possibly indicating a shift in the economic environment. By placing more weight on more recent data, *constant-gain learning* (CGL) provides a natural model of this potential alertness.

To illustrate constant-gain learning in the presence of structural change, the cobweb model is modified to include a proportional sales tax $\tau$: whence if $p_t^e$ is the expected market price then expected marginal (net) revenue is $(1-\tau)p_t^e$, yielding the TE equation

$$p_t = \mu + \alpha(1-\tau)p_t^e + \delta' w_{t-1} + \eta_t.$$

Figure 1 illustrates learning dynamics for an unanticipated, permanent (and admittedly large) tax increase of 50% in period 500. The upper panels provide the dynamics of beliefs $(a_t, b_t)$, which were initialized in REE. The lower-left panel shows the price dynamics over the

[18] This holds also for the decreasing gain case for appropriate assumptions on the gain sequence, including $\gamma_t = t^{-1}$, where instead $\tau = \sum_{i=1}^{t} \gamma_i$.

whole simulation and the lower-right panel shows the price dynamics (black) in a 100-period window centered at the time of the tax hike; also included in this panel, for comparison purposes, are the price dynamics under REE (red). All horizontal, dashed lines correspond to REE means.

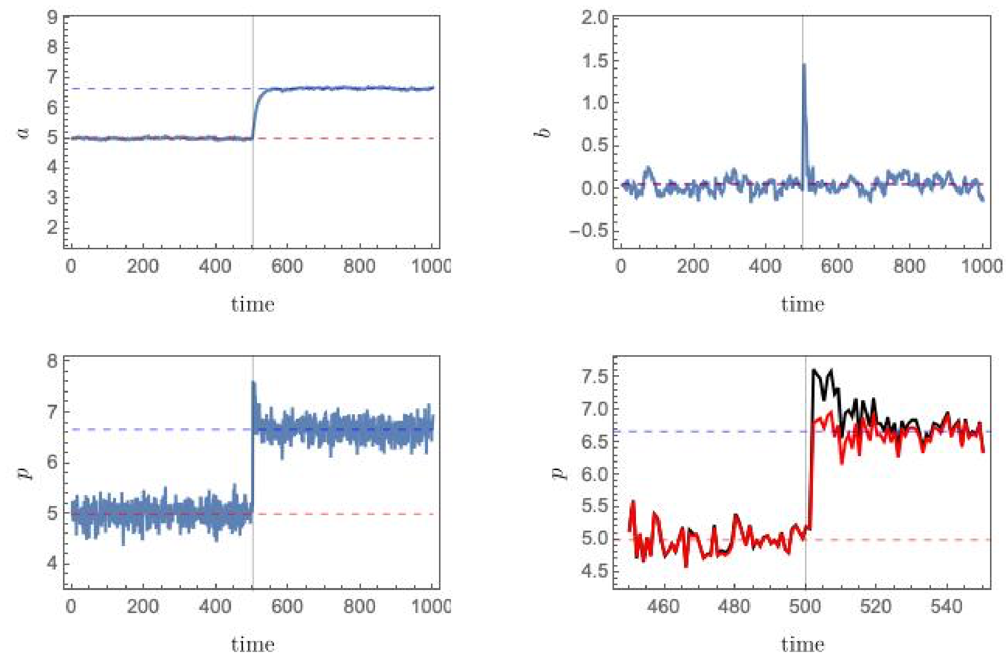


Figure 1: Cobweb model with tax increase.

Note that, prior to the shock, beliefs stay near the REE values, and, as indicted in the lower-right panel, the learning and REE price paths are very nearly identical. The tax hike in period 500 causes a sharp decrease in supply, resulting in a price spike under both learning and RE. The REE price path simply stays aligned with the new REE price process; the learning path, however, is more complex. The CGL algorithm initially attributes the price spike to an increased responsiveness of prices to $w$, as seen in the upper right-panel. This misperception causes further curtailment of supply, resulting in an over-shooting of the price path. Newly generated data quickly disabuse agents of their erroneous beliefs, as the constant term rises to its new steady-state level – see upper-left panel – and the learning price path returns to match its REE analog.

Even in stationary environments CGL can provide interesting dynamics over-and-above those observed under rational expectations. For example, the heavier weight placed on newly observed data can induce excess volatility, particular when the model's expectational feedback is strong. This can again be illustrated using the cobweb model. When $\delta = 0$ and the PLM is simply given by a constant, the excess volatility, defined as the unconditional variance of price under learning relative to its variance under REE, can be derived analytically, and in doing so it can be shown that the learning dynamics are stationary provided $\alpha \in (1 - 2/\gamma, 1)$.

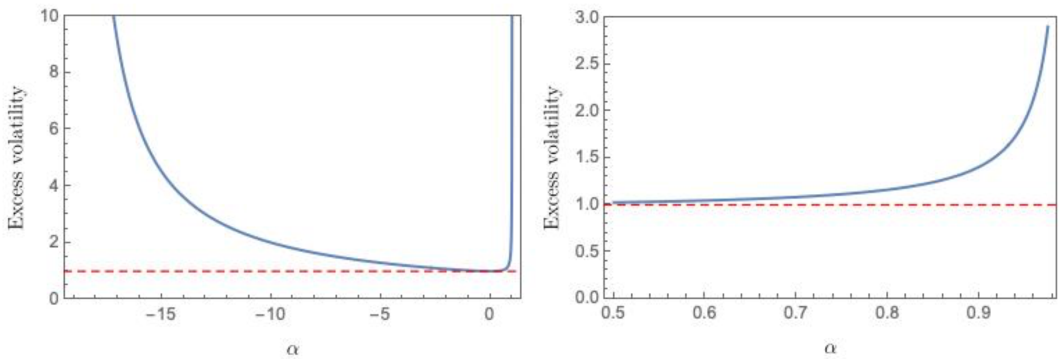


Figure 2: Excess volatility under constant gain learning

Figure 2 provides the relevant plot, allowing the expectational feedback parameter $\alpha$ to vary: the left panel provides the full range of $\alpha$ and the right panel provides a closer look for $\alpha \in (0.5, 1)$.[19] Notice that for values of $\alpha$ near $1 - 2/\gamma$ or one the excess volatility can be arbitrarily large. Simulations show that the same pattern emerges for examined calibrations with $\delta > 0$ and in which agents' PLM includes the shock $w$ as a regressor.

In simple asset-pricing models, with strong positive expectational feedback, constant gain learning can lead to substantial excess volatility of asset prices, in line with the above theoretical results and the well-known empirical results of Shiller (1981). See, for example, Bullard, Evans, and Honkapohja (2009). Constant gain learning also may be able to explain some well-known empirical puzzles in other areas of asset-pricing, including foreign exchange rates and the yield curve: see Chakraborty and Evans (2008) and Sinha (2016).

## *Heterogeneous expectations and multiple forecasting models*

Although the focus of this chapter has been on representative-agent models in which agents have homogeneous expectations, the AL approach can be extended to incorporate heterogeneous expectations in a variety of ways. Evans, Honkapohja, and Marimon (2001) show how it is possible to allow for random inertia and heterogeneous gains across agents in updating expectations.

Some authors have allowed for heterogeneous expectations by assigning a fixed proportion of agents to different forecasting models, see Branch and McGough (2004), Adam (2005) and Berardi (2007). Another way to incorporate heterogeneous expectations is to assume that several distinct forecasting models are available and to use discrete-choice (or

[19] Here the gain is chosen as $\gamma = 0.1$.

"dynamic predictor selection") models to determine the proportion of agents using each type of model based on a measure of fitness (e.g. estimated mean-square error) and an intensity of choice parameter. This approach was developed in Brock and Hommes (1997) and is a major focus of Hommes (2013), in which the set of forecast rules available are usually taken to be of the type described in the Section titled *Behavioral Rules*. For an application of the discrete-choice approach to survey data see Branch (2004). See also Pfajfar and Zakelj (2014).

A distinct but related approach uses replicator dynamics to model the evolution of the proportions of a set of forecast rules over time. For examples see Guse (2010) and Branch and McGough (2008). Another AL approach is to allow for two types of agents, in which one group of agents draws on the forecasts of other agents. See Granato, Guse, and Wong (2008).

Genetic-algorithm learning (or "social learning"), first applied in economics by Arifovic (1994), allows by design for considerable heterogeneity of expectations and decisions. Applications include Arifovic (1996), who studies foreign exchange-rate dynamics, and the Arifovic, Bullard, and Duffy (1997) examination of economic growth and development. For recent applications to monetary economics see Arifovic, Bullard, and Kostyshyna (2013), Arifovic, Schmitt-Grohe, and Uribe (2018) and Arifovic, Grimaud, Salle, and Vermandel (2020).

Branch and Evans (2006a) extend the dynamic predictor selection methodology of Brock and Hommes (1997) with the LS-learning AL approach. Agents are assumed to choose between two or more forecasting models that are underparameterized, e.g. in their excluding different relevant variables. Coefficient parameters of each model are updated over time in accordance with LS-learning, and the proportions of agents using each model are based on dynamic predictor selection and a model of fitness. Branch and Evans (2006a) show existence of a "misspecification equilibrium," in which agents forecast optimally given their choices, with forecast model parameters and predictor proportions determined endogenously. When there is negative expectational feedback there can be intrinsic heterogeneity, in which multiple forecast rules across agents are employed. When there is positive feedback, as in Branch and Evans (2007), there can exist multiple stable misspecification equilibria, and pooling and regime-switching may occur endogenously under learning.

When multiple forecasting models are on the table, another approach is to assume that agents use econometric methods to choose between them. In the context of foreign exchange rates, Markiewicz (2012) develops a model learning approach, based on the BIC model selection criterion, and shows that it can explain a change in the GBP/USD exchange rate volatility. A general analysis of the model validation approach is given in Cho and Kasa (2015).

When a discrete number of alternative forecasting rules are under consideration, an alternative approach is that individual agents may combine them, e.g. using Bayesian model-averaging. This approach is explored in Gibbs (2017), where among other results it is shown that multiple equilibria can arise. Bayesian model averaging is used in Evans, Honkapohja, Sargent, and Williams (2013) in the context of the cobweb model. Two models are under consideration by agents: one nests the REE, whereas the other model incorrectly assumes that a key parameter is subject to random-walk drift. They show that in the positive feedback case it is possible for Bayesian model averaging to lead agents select the misspecified model. Using large-deviation tools Cho and Kasa (2017) develop this possibility further in the context of a simple asset-pricing model.

# Multivariate Linear Models

While the cobweb model (2) was convenient for introducing adaptive learning concepts and tools, macroeconomic models usually have more complex dynamic structures. Typically these include both forward-looking and backward-looking endogenous variables. For example, the risk-neutral asset-pricing model takes the form

$$p_t = \beta E_t^* p_{t+1} + d_t,$$

where $p_t$ is the price of an equity and $d_t$ is the dividend, assumed following an exogenous stochastic process. The same equation form arises for simple PPP models of the foreign exchange rate and for special cases of the overlapping generations model of money. In many models costs of adjustment are also important. For example, Sargent's linear-quadratic version of the Lucas-Prescott model of investment takes the form

$$k_t = \alpha + \beta E_t^* k_{t+1} + \delta k_{t-1} + \gamma u_t,$$

where $k_t$ is the industry capital stock and $u_t$ is exogenous. In this case, as is typically true of serious multivariate macroeconomic models, the key endogenous variables are both forward and backward looking.

An important new feature that can arise in the context of forward-looking linear models is the possibility of *indeterminacy*, i.e. the existence of multiple stationary REE. A linear model is said to be *determinate* if there is a unique nonexplosive RE solution.

Most dynamic stochastic general equilibrium (DSGE) macro models are, in addition, nonlinear, but they are frequently analyzed under RE using a linearization around a steady state, and typically take the standard form

$$y_t = M E_t y_{t+1} + N y_{t-1} + P v_t, \tag{21}$$

$$v_t = F v_{t-1} + \varepsilon_t, \tag{22}$$

where $y_t$ is a vector of endogenous variables, in deviation from mean form, and $v_t$ is an exogenous, observable, stationary VAR(1) process driven by the white noise shock $\varepsilon_t$. The usual *minimal state variable* (MSV) RE solution takes the form

$$y_t = \bar{a} + \bar{b} y_{t-1} + \bar{c} v_t, \tag{23}$$

with here $\bar{a} = 0$.

## *E-stability in multivariate linear models*

From the adaptive learning perspective, the ideal would be to build up every economic model from the agent level, with explicit assumptions about how agents formulate their decisions rules and make forecasts in nonlinear infinite or long-horizon settings. A convenient short-cut that can often be very informative, is to simply start from the linearized RE model and replace $E_t y_{t+1}$ by one-step ahead forecasts $E_t^* y_{t+1}$. This is the reduced-form learning approach. Under adaptive learning agents believe that $y_t = a + by_{t-1} + cv_t$ and use LS learning to update over time their estimates $(a_t, b_t, c_t)$ of the parameters.

The E-stability principle can be used to evaluate stability of the RE solutions to (21)-(22) under learning. The analysis for an MSV solution proceeds as follows. For the PLM

$$y_t = a + by_{t-1} + cv_t$$

the corresponding forecasts are $E_t^* y_{t+1} = (I+b)a + b^2 y_{t-1} + (bc + cF)v_t$. As noted above, $v_t$ is assumed observable,[20] and here, for convenience and without loss of generality, $F$ is assumed known. Inserting the forecasts into the model yields the ALM

$$y_t = M(I+b)a + (Mb^2 + N)y_{t-1} + (Mbc + NcF + P)v_t,$$

which gives the mapping from PLM to ALM:

$$T(a, b, c) = (M(I+b)a, Mb^2 + N, Mbc + NcF + P).$$

The REE $(\bar{a}, \bar{b}, \bar{c})$ is a fixed point of $T(a, b, c)$. If the E-stability ODE

$$d/d\tau(a, b, c) = T(a, b, c) - (a, b, c)$$

is (locally asymptotically) stable at the REE it is said to be E-stable. Evans and Honkapohja (2001), Chapter 10, show that the corresponding E-stability conditions can be stated in terms of the derivatives

$$DT_a = M(I + \bar{b}),\ DT_b = \bar{b}' \otimes M + I \otimes M\bar{b},\ DT_c = F' \otimes M + I \otimes M\bar{b}, \tag{24}$$

where $\otimes$ denotes the Kronecker product. If all eigenvalues of these matrices have real parts less than one then the REE is E-stable.

For a given numerical values of $M, N, P, F$ it is straightforward to compute the MSV solutions, to determine whether the model is determinate or indeterminate, and to use (24) to assess each MSV solution for local stability under learning. In an indeterminate model with multiple stationary MSV solutions, E-stability can be used as a selection criterion. For some applications general analytical results can be obtained for determinacy and E-stability. The ease with which E-stability conditions can be assessed numerically is an advantage of the reduced-form short-cut to assessing stability under adaptive learning in linearized models.

[20] Bullard and Singh (2012) consider the case of an unobserved exogenous regime-switching process for technology with agents using Bayesian techniques to learn the latent state.

### *Example: monetary policy*

An early application of E-stability to New Keynesian (NK) models, using the reduced-form adaptive learning approach, was the assessment by Bullard and Mitra (2002) of alternative interest-rate rules by monetary policymakers. The basic linearized NK model (see e.g. Clarida, Gali, and Gertler (1999)) takes the form

$$x_t = -\varphi\left(i_t - E_t^*\pi_{t+1}\right) + E_t^* x_{t+1} + v_{xt} \quad (25)$$

$$\pi_t = \beta E_t^*\pi_{t+1} + \lambda x_t + v_{\pi t}, \quad (26)$$

where $0 < \beta < 1$ is the discount factor and $\varphi, \lambda > 0$. Here $x_t$ is the output gap, $\pi_t$ is inflation and $i_t$ is the nominal interest rate, all of which are expressed in deviation from their steady-state values.[21] These equations are often called the NK IS and NK PC equations. The exogenous IS and inflation shocks $v_{xt}, v_{\pi t}$ are typically assumed to be independent, stationary AR(1) processes. These model equations must be supplemented by a monetary policy rule. Typically this takes the form of a "Taylor" rule, which sets the interest rate in response to either contemporaneous, lagged or expected inflation and output. Since monetary policy is often treated as forward-looking a simple such rule takes the form

$$i_t = \alpha_\pi E_t^*\pi_{t+1} + \alpha_x E_t^* x_{t+1}, \quad (27)$$

where $\alpha_\pi, \alpha_x > 0$. Sometimes an exogenous shock is also included in the interest-rate setting equation. Alternative rules are current-data rules

$$i_t = \alpha_\pi \pi_t + \alpha_x x_t, \quad (28)$$

and backward-looking rules,

$$i_t = \alpha_\pi \pi_{t-1} + \alpha_x x_{t-1}. \quad (29)$$

The current-data rule (28) has been criticized on the grounds that, first, policymakers in fact respond mainly to the expected state of the economy in the near future, and, second, accurate information on current output and inflation is only available with a lag.

Bullard and Mitra (2002) assess the NK model for determinacy, i.e. whether there is a unique non-explosive RE solution, and for stability under learning, which is examined using E-stability for the three alternative policy rules and for different choices of $\alpha_\pi, \alpha_x$. Determinacy can be assessed by putting the system in standard first-order matrix form and comparing the number of eigenvalues outside the unit circle to the number of "free" (current endogenous) variables.[22] The E-stability conditions can be used to assess stability

[21] For this to be a suitable approximation the (steady state) target inflation rate for the monetary authorities needs to be close to zero.

[22] If these are equal the model is determinate. If there are fewer eigenvalues outside the unit circle, the model is indeterminate: there are multiple stationary REE. For details see, e.g., Blanchard and Kahn (1980) and Sims (2001).

of an REE under adaptive learning. These are distinct conditions. For the NK model (25)-(26), with either policy rule (27) or (28), inserting the policy equation leads to a bivariate forward-looking model[23] of the form

$$y_t = ME_t y_{t+1} + Pv_t, \tag{30}$$

where $y_t' = (x_t, \pi_t)$ and $v_t' = (v_{xt}, v_{\pi t})$, and where $M$ and $P$ are functions of the structural parameters. In this case the model is determinate if both roots of $M$ lie inside the unit circle, while if one or both roots lie outside the unit circle the model is indeterminate. When the model is determinate the unique stationary solution takes the form

$$y_t = \bar{c}v_t$$

for suitable $2 \times 2$ matrix $\bar{c}$. When the model is indeterminate there continues to be an MSV solution of this form, but there are also other stationary solutions.

E-stability of the MSV solution to (30) is also straightforward to compute. Because $\bar{b} = 0$ in the MSV solution in the current cases, the PLM estimated by agents takes the form

$$y_t = a + cv_t.$$

Note that intercepts are included in the PLM because agents will generally need to estimate the means of the $x_t$ and $\pi_t$ as well as the impact of $v_t$. The E-stability conditions reduce to checking that all eigenvalues of

$$DT_a = M \text{ and } DT_c = F' \otimes M$$

have real parts less than one.

Bullard and Mitra show that for the rule (28), both determinacy and E-stability hold provided $\lambda(\alpha_\pi - 1) + (1-\beta)\alpha_x > 0$. In particular, if the "Taylor principle" $\alpha_\pi > 1$ is satisfied then the model is determinate and the MSV solution is stable under learning. However, matters are more subtle in the case of forward-looking and backward-looking rules, and Bullard and Mitra (2002) computed the results for standard calibrations of $\varphi, \beta, \lambda$.

For the policy rule (27), if $\alpha_\pi > 1$, but not too large, and with $\alpha_x > 0$ small, then both determinacy and E-stability hold. However, there is a large region with $\alpha_\pi > 1$ and $\alpha_x > 0$ relatively large in which the model is indeterminate and the MSV solution is stable under learning. Of course in the indeterminate case other stationary REE exist as solutions. Subsequent research by Honkapohja and Mitra (2004) and Evans and McGough (2005b) established that in this case there exist "sunspot solutions," i.e. solutions that in addition depend on extraneous variables, which are stable also stable under learning. The Section titled *Multiple Equilibria: Sunspots* discusses sunspot equilibria more generally.

[23] These findings assume the model is complete. A debt-dynamics equation would need to be included if there is "active" fiscal policy.

Finally, for the policy rule (29) the MSV solution instead takes the form (23). Bullard and Mitra found that for $\alpha_\pi > 1$ and $\alpha_x > 0$ small the MSV solution is determinate and E-stable. However for $\alpha_\pi > 1$ and $\alpha_x > 0$ large the REE solutions are explosive and for $\alpha_\pi < 1$ and $\alpha_x > 0$ for an intermediate range of values, the model is determinate, but the stationary MSV solution is not E-stable. This last result is an illustration of the fact that, while in a wide range of models determinacy implies stability under learning, this is not a fully general result.[24]

There is an extensive literature on monetary policy and adaptive learning within the linear framework (25)-(26) and in hybrid versions that allow for habit formation and indexation that bring in lagged endogenous variables. This literature includes alternative implementations of optimal policy, simultaneous estimation of structural parameters by monetary policymakers, policymaker uncertainty about structural parameters, implications of constant-gain learning and/or misspecified PLMs, policy allowing for expectational heterogeneity, and the relative desirability of price level vs. inflation targeting.[25] See the survey papers Evans and Honkapohja (2003a), Evans and Honkapohja (2009a) and Evans and Honkapohja (2013). For a recent and comprehensive survey of monetary policy under learning that emphasizes the long-horizon "anticipated utility" approach, see Eusepi and Preston (2018b).

In the aftermath of the Great Recession, a major policy issue in the US, beginning around 2014, was normalization of monetary policy: how quickly should interest rates be returned to normal levels from the near-zero rates that extended from January 2009 through December 2015? A neo-Fisherian view, advanced for example by Cochrane (2015), argued for immediately increasing the policy rate to the level consistent with the targeted steady-state inflation rate and fixing the policy rate at that level. Rational expectations, and uniqueness of the steady state under an interest-rate peg, would appear to guarantee that inflation would then return to its targeted level. Evans and McGough (2018b) and Evans and McGough (2018a) argued that under AL such a policy would lead to unstable paths and which could, in particular, lead to renewed recession. Complementary arguments are provided by Garcia-Schmidt and Woodford (2019). The argument that fixed interest-rate pegs have undesirable properties under learning can be traced back to Howitt (1992).

# Multiple Equilibria: Sunspots

A series of papers including Shell (1977), Azariadis (1981), Cass and Shell (1983), Grandmont (1985) and Azariadis and Guesnerie (1986) established that in simple nonstochastic nonlinear

[24] For further discussion of this issue see McCallum (2007) and Bullard and Eusepi (2014).

[25] The possibility that some implementations of optimal policy may not be stable under AL was emphasized by Evans and Honkapohja (2003c) and Evans and Honkapohja (2006). For optimial policy rules that are stable under AL and allow for structural parameter uncertainty see Evans and McGough (2007). McGough, Rudebusch, and Williams (2005) examine monetary policy conducted using long rates. Honkapohja and Mitra (2019) study price-level targeting when when agents update their assessment over time of the credibility of the policy.

overlapping generations models of money there can exist multiple REE taking the form of stationary sunspot equilibria (SSEs) or regular periodic cycles. Here the term "sunspot" refers to an exogenous stochastic process that is "extrinsic" in the sense that is unconnected to fundamental shocks (the latter, also called "intrinsic" shocks, include taste, productivity, and other relevant random shocks). The SSEs shown to exist typically took the form of 2-state Markov processes.

Nonstochastic nonlinear models can also exhibit multiple steady states, and these can have SSEs taking values near different steady states. More recently RBC-type models with nonconvexities arising from externalities, increasing returns and monopolistic competition, distortionary taxes, etc., have been developed that possess an *indeterminate* steady state, in which there exist SSEs local to the indeterminate steady state. As already noted the indeterminate case can arise in New Keynesian (NK) models for certain specification of the interest-rate rule, which implies the existence of SSEs. Finally, a recent strand of the asset-price bubbles literature emphasizes cases in which there are nonexplosive asset-bubbles, and these may be viewed as SSEs.

An advantage of the AL approach is that it provides a way of assessing the plausibility of SSEs: if agents have PLMs that condition on an observed sunspot process, will the SSE be (at least locally) stable under leaning? The seminal paper by Woodford (1990) showed that SSEs in an overlapping-generations model could indeed be stable under AL. The work of Evans (1989), Evans and Honkapohja (1994), Evans and Honkapohja (2003b) and others showed that SSEs in general may or may not be stable under learning, and that stability can be readily assessed using the E-stability principle.

While much of the early work focused on simple nonlinear models, the existence of indeterminate multivariate RBC-type models and NK models has shown how the E-stability tools can be extended to examine SSEs in linearized models. The discussion is developed in this context, focusing on two special cases: (i) a purely forward-looking univariate or multivariate model, and (ii) a forward and backward-looking univariate model.

## *SSEs in a forward-looking model*

Start with the simplest possible case of a univariate purely forward-looking linear model

$$y_t = \beta E_t^* y_{t+1}, \tag{31}$$

where $\beta \neq 0$, where for simplicity the intercept is normalized to zero and exogenous shocks are omitted. Since the underlying economic model has not been specified the reduced-form learning viewpoint is adopted. Provided $|\beta| < 1$ there is a unique nonexplosive REE given by the MSV solution

$$y_t \equiv 0, \tag{32}$$

all $t$. For the PLM $y_t = a$ the corresponding ALM is $y_t = \beta a$ and it follows that the REE (32) is E-stable and hence stable under AL if $\beta < 1$.

If instead $|\beta| > 1$ then (32) remains a solution, and for $\beta < -1$ it remains stable under AL for the PLM $y_t = a$. However, in this case there are other stationary solutions. In particular there are solutions of the form

$$y_t = \beta^{-1} y_{t-1} + \varepsilon_t, \tag{33}$$

where $\varepsilon_t$ is, for example, an *iid* exogenous observable white-noise shock. Here $\varepsilon_t$ is an extraneous variable, often called a sunspot, and the solutions (33) are called sunspot solutions. It turns out these sunspot solutions are not stable under AL for PLMs of the form $y_t = a + by_{t-1} + \varepsilon_t$.[26]

However, there is another representation of SSEs that *can* be stable under AL. For the case $|\beta| > 1$ consider the SSE solutions

$$\begin{aligned} y_t &= \eta_t \text{ where} \\ \eta_t &= \beta^{-1}\eta_{t-1} + \varepsilon_t. \end{aligned}$$

Here the sunspot $\eta_t$ is a stationary AR(1) process, assumed observable.[27] Consider now PLMs of the form

$$y_t = a + b\eta_t. \tag{34}$$

The $T-$map is $T(a, b) = (\beta a, b)$ and the eigenvalues of $DT$ are 1 and $\beta$. The eigenvalue of 1 is a reflection of the fact that in the linear set-up there is actually a continuum of SSE depending on $\eta_t$, i.e. $y_t = \eta_t'$ is also an SSE if $\eta_t'$ is a scalar multiple of $\eta_t$.[28] It follows that if $\beta < -1$ the set of sunspots of the form (34) is E-stable[29] and hence stable under AL, whereas for $\beta > 1$ SSEs are not stable under AL.

This approach extends to multivariate frameworks including those with intrinsic shocks, i.e. to models of the form

$$\begin{aligned} y_t &= ME_t^* y_{t+1} + Pv_t, \\ v_t &= Fv_{t-1} + \tilde{v}_t. \end{aligned}$$

A special case is the standard bivariate NK model with forward-looking interest-rate rule. Here $v_t$ is a vector of observable stationary exogenous shocks and the roots of $F$ are assumed inside the unit circle. If the roots of $M$ are inside the unit circle the model is determinate and there is as unique stationary solution, taking the MSV form $y_t = \bar{c}v_t$. In the indeterminate

[26] The sunspot representations (33) are sometimes called "general-form" representations, and (34) are then called "common-factor" represeentations.

[27] $\varepsilon_t$ is in general a martingale difference sequence, which of course includes *iid* continuously measured white noise processes. However also allowable are finite-state processes that generate finite-state Markov processes for $\eta_t$.

[28] This is an artifact of the linear set-up. Also the AR(1) coefficient $\beta^{-1}$ in (34) is sometimes called the "resonance-frequency." This too is an artifact of the linear specification. In nonlinear models an interval of suitable coefficients is consistent with sunspot equilbria.

[29] Informally, a set is stable if trajectories initialized near the set converge to some point inside the set.

case, in which one or more roots of $M$ are outside the unit circle there are SSEs taking the form

$$y_t = \bar{c}v_t + \bar{d}\zeta_t$$

for suitable exogenous sunspot processes $\zeta_t$. For example if $M$ has one eigenvalue $\phi$ with magnitude greater than one, then $\zeta_t$ could be a sunspot represented as an AR(1) process with damping coefficient $\phi^{-1}$.

As noted in the Section titled *Example: monetary policy*, the NK model with forward-looking interest-rate rules can lead to SSEs that are stable under AL. See Honkapohja and Mitra (2004) and Evans and McGough (2005b) for further discussion.

## SSEs with predetermined variables

Consider now a univariate model taking the form

$$y_t = \beta E_t^* y_{t+1} + \delta y_{t-1} + v_t, \tag{35}$$

where for simplicity an intercept is omitted and $v_t$ is taken to be a white noise exogenous process. To keep the analysis simple and generic we continue to take a reduced-form approach to AL in this set-up. Assume $\beta \neq 0, \delta \neq 1$ and $\beta + \delta \neq 1$.

Associated with this model is the quadratic $\beta b^2 - b + \delta = 0$ and attention is restricted to the case in which the roots $b_1, b_2$ are real. The model is determinate if exactly one of the roots is smaller than one in magnitude and one is larger than one in magnitude. In this case the unique stationary solution takes the MSV form

$$y_t = b_1 y_{t-1} + (1 - \beta b_1)^{-1} v_t$$

where $|b_1| < 1 < |b_2|$.[30] For PLMs of the form $y_t = a + by_{t-1} + kv_t$ it is straightforward to work out the E-stability conditions and it can be shown that in the determinate case the MSV solution above is E-stable.[31]

In the indeterminate case there are *common-factor* sunspot solutions, depending on a stationary sunspot $\zeta_t$, that take the form

$$\begin{aligned} y_t &= b_1 y_{t-1} + d\zeta_t + (1 - \beta b_1)^{-1} v_t, \text{ where} \\ \zeta_t &= b_2 \zeta_{t-1} + \varepsilon_t \end{aligned}$$

for a martingale difference sequence $\varepsilon_t$. Furthermore, it can be shown that there are regions of the parameter space for which these sunspot solutions are E-stable and stable under AL. For a systematic treatment of the stability of SSEs in this model (35) see Evans and McGough (2005c). For additional applications see Evans and McGough (2005b) and Ellison and Pearlman (2011).

[30] Here $b_1 = (2\beta)^{-1}\left(1 - \sqrt{1 - 4\beta\delta}\right)$ and $b_2 = (2\beta)^{-1}\left(1 + \sqrt{1 - 4\beta\delta}\right)$.

[31] The precise information set needs to be specified. A common assumption is that $y_{t-1}$ and $v_t$ are in the time $t$ information set, but that $y_t$ itself is not observed when $E_t^* y_{t+1}$ is formed. Including $y_t$ in the time $t$ information set can in some cases alter the E-stability conditions.

## *Discussion*

In his IMF Blog of Dec. 11, 2011, Olivier Blanchard stated that *" ... the world economy is pregnant with multiple equilibria – self-fulfilling outcomes of pessimism or optimism, with major macroeconomic implications."* This view makes imperative understanding when and how sunspot equilibria, which represent and characterize the class of stationary multiple equilibria, are consistent with the modern DSGE paradigm. Studying the existence and attainability of SSEs, assessed by local stability under AL, are natural tools for assessing the importance of this perspective.

Indeed, considerable work has been done exploring SSEs and their stability under AL in a variety of modern settings, including linearized RBC-type models with nonconvexities, NK models under alternative policy rules, endogenous growth frameworks, and asset-pricing models. Assessments of SSEs in indeterminate RBC-type models are given in Evans and McGough (2005a), Duffy and Xiao (2007) and McGough, Meng, and Xue (2013). Evans, Honkapohja, and Romer (1998) provide an example of stable SSEs in an endogenous growth model. Evans, Honkapohja, and Marimon (2007) show existence of stable SSEs in a cash-in-advance monetary model with seigniorage- and tax-financed government spending. Zanna (2009) shows existence of stable SSEs in a class of small open economy models. Existence and stability under learning of SSEs in regime-switching models has been examined by Branch, Davig, and McGough (2013). Shea (2013) finds stable SSEs in a model of learning by doing with short-sighted managers. For an example of the new generation of bubble models, with stochastically stationary bubbles, Miao, Shen, and Wang (2019) provide an assessment of stability under learning of the alternative solutions and a corresponding reassessment of the policy implications. Branch, McGough, and Zhu (2019) show that stable "statistical sunspot" equilibria can exist in models with a unique REE if agents use misspecified forecasting models.

There are also several experimental papers that look at the attainability of SSEs in laboratory settings. See, in particular, Marimon, Spear, and Sunder (1993), Duffy and Fisher (2005) and Arifovic, Evans, and Kostyshyna (2020).[32] Finally, recent theoretical research show that existence and assessment of near-rational SSEs can proceed in a systematic way in nonlinear general equilibrium settings using standard tools. In particular Evans and McGough (2020c) show that indeterminacy and stability of the MSV solution in the linearized model imply the existence of stable sunspot equilibria in the linearized model, and stable near-rational sunspot equilibria in the nonlinear model. These findings are collectively referred to as the *MSV Principle*.

[32] Arifovic, Hommes, and Salle (2019) study the stability of perfect-foresight cycles in an experimental setting.

# Other Applications and Extensions

The implications of the adaptive learning approach for macroeconomics are potentially major and wide-ranging. A large body of work has aimed to implement AL in calibrated or estimated models, as well as to experimental and survey data. In addition to the applications discussed in the Introduction, other important contributions includes the following.

Orphanides and Williams (2007) show that efficient monetary policies that take account of AL by private agents, and misperceptions of natural rates by policymakers, indicate the need for greater policy inertia, a larger response to inflation, and a smaller response to the perceived unemployment gap than would be optimal under RE. Furthermore policies that would be optimal under RE can perform poorly under AL. The implications of incorporating AL in applied medium-scale DSGE models has been explored in Slobodyan and Wouters (2012b) and Slobodyan and Wouters (2012a). The latter paper models agents as using small forecasting models. The estimated model is used to explain a decline in the mean and volatility of inflation, and the results are linked to survey evidence on inflation expectations. See also Elias (2020). Aguilar and Vazquez (2019) explore the role of the term structure in estimated DSGE models with learning. Using survey data on expectations as well as aggregate macro data, Milani (2011) estimates both the structural parameters of a macro model and expectations shocks, and concludes that expectation shocks can account for about half of business cycle fluctuations. Eusepi, Moench, Preston, and de Carvalho (2020) show how to use AL in a Rotemberg-pricing model to match low-frequency movements in inflation expectations.

At mentioned in the Introduction, using a long-horizon AL real business cycle model calibrated to US data, 1948:I to 2007:IV, Eusepi and Preston (2011) show that the AL model dynamics have several important differences from RE. For example, a smaller productivity innovation variance is needed to fit output volatility, and under AL the IRFs show hump-shaped patterns corresponding to initial overoptimism about future returns to investment and overpessimism concerning future wages. In addition the expectation forecast errors exhibit some features found also in the Survey of Professional Forecasters. Additional work includes Kuang and Mitra (2016), who stress the business cycle implications of expectations and imperfect knowledge of long-run growth rates.[33] Fiscal policy under AL in RBC-type models has been studied in Giannitsarou (2006), Evans, Honkapohja, and Mitra (2009), Mitra, Evans, and Honkapohja (2013) and Mitra, Evans, and Honkapohja (2019).

A number of papers have used AL to address various aspects related to macroeconomic policymaking. Cogley and Sargent (2005), Bullard and Eusepi (2005), Primiceri (2006) and Sargent, Williams, and Zha (2006) address the rise and fall of inflation in the US over the 1960-1990 period. For a cross-country study of disinflations see Gibbs and Kulish (2017). Income distribution dynamics in an incomplete markets model with AL is studied in Giusto (2014). Several papers have examined whether or not Ricardian Equivalence holds

[33] Dombeck (2020) studies stability under AL in a news-shock model.

under AL. Agents may or may not be assumed to impose the government's intertemporal budget constraint in their consumption function, and results also depend on how agents make forecasts of key aggregates. Papers that examine these issues include Eusepi and Preston (2010), Evans, Honkapohja, and Mitra (2012), Woodford (2013), Benhabib, Evans, and Honkapohja (2014) and Branch and Gasteiger (2019).

Other applications to financial markets besides those mentioned in the Introduction include Branch and Evans (2010) who show how AL can extend the range of asset-price dynamics, and Adam, Marcet, and Beutel (2017), who adopt the "internal rationality" approach. Benhabib and Dave (2014) and Elias (2016) address the impact of learning and expectations shocks on asset price dynamics. Nagel and Xu (2019) stress the fading memory interpretation of non-decreasing gain learning in empirical asset-pricing models. Kuang (2014) and Gelain and Lansing (2014) focus, respectively, on housing and credit cycles and the house-price to rent ratio. Branch, Petrosky-Nadeau, and Rocheteau (2016) and Pintus and Suda (2019) consider business-cycle models with collateral constraints. Evans, Hommes, McGough, and Salle (2019) provide laboratory experimental results showing the impact on asset price dynamics of the forecast and decision-horizons of subjects.

Another major policy issue, particularly since the Great Recession, has been the role of the zero lower bound (ZLB). Benhabib, Schmitt-Grohe, and Uribe (2001) showed that under RE a global Taylor-type interest-rate rule that is active at the targeted steady state implies the existence of a second (indeterminate) steady state at a lower rate of inflation (or deflation). Using Euler-equation learning in a nonlinear model, Evans, Guse, and Honkapohja (2008) showed that while the targeted steady state is locally stable under AL, the unintended low-inflation steady state is unstable under learning. However, they argue that the global dynamics reveal the potential for major recessions if there is a sufficiently large pessimistic shock to output and inflations expectations. If the negative expectations shock is large enough the economy enters a deflationary trap region in which output and inflation fall over time with interest rates at the ZLB. However, fiscal policy can be effective in this situation. These results are extended in Benhabib, Evans, and Honkapohja (2014), using a long-horizon model, which shows that aggressive fiscal policy can be effective in escaping the deflation trap even if agents take full account of the tax consequences of fiscal policy. Evans, Honkapohja, and Mitra (2020) develop a global nonlinear stochastic New Keynesian model, with three steady states and a stagnation regime underpinned by the low-level steady state. A large fiscal stimulus may be needed to avoid stagnation and the impacts of forward guidance, policy delay, credit frictions and central bank credibility are examined.

Models in which agents are engaged in adaptive learning also raise new econometric issues concerning identification and the asymptotic distribution for the "external" estimation problem of economists making inferences from the data concerning structural parameters of the model. These issues are discussed in Chevillon, Massmann, and Mavroeidis (2010) and Chritopeit and Massmann (2018).

# The Reach of Adaptive Learning

Adaptive learning in macroeconomics is an active field of research with a wide diversity of approaches. The AL approach can be applied to virtually any macroeconomic model in which household and business forecasts play an important role and optimal dynamic decision-making is central. In principle the AL approach can be fully grounded in agent-level decision-making, with explicit aggregation. However, various analytical tools and short-cuts can also provide key results, and the AL approach is well-suited for incorporation into computational models and numerical simulations.

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