Stable Finite-State Markov Sunspots*

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Abstract

We consider a linear univariate rational expectations model, with a predetermined variable, and study existence and stability of solutions driven by an extraneous finite-state Markov process. We show that when the model is indeterminate there exists a new class of k-state dependent sunspot equilibria in addition to the k-state sunspot equilibria (k-SSEs) already known to exist in part of the indeterminacy region. The new type of equilibria, which we call ergodic k-SSEs, are driven by a finite-state sunspot but can have an infinite range of values even in the nonstochastic model. Stability under econometric learning is analyzed using representations that nest both types of equilibria. 2-SSEs and ergodic 2-SSEs are learnable for parameters in proper subsets of the regions of their existence. Our results extend to models with intrinsic random shocks.

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1 Introduction

The existence of "self-fulfilling" solutions, driven by extraneous stochastic processes known as "sunspots," was initially demonstrated by (Shell 1977), (Azariadis 1981), (Cass and Shell 1983), (Azariadis and Guesnerie 1986) and (Guesnerie 1986) in simple stylized models, such as the Overlapping Generations model of money. More recently the existence of such solutions in linearized versions of Real Business Cycle models with distortions has emphasized the possibility that sunspot equilibria may provide a way of accounting for macroeconomic fluctuations. For the recent literature see (Guesnerie and Woodford 1992), (Farmer 1999) and (Benhabib and Farmer 1999).

A question that has arisen in this literature concerns the attainability of sunspot equilibria. That sunspot solutions could be stable under adaptive learning was demonstrated for the basic Overlapping Generations model by (Woodford 1990), and conditions for local stability under adaptive learning were provided in (Evans and Honkapohja 1994b) and (Evans and Honkapohja 2003) for one-step forward looking univariate nonlinear models.[1] The solutions considered in these papers take the form of a finite-state Markov process, a type of solution that is prominent in the theoretical literature and described at length, for example, in (Chiappori, Geoffard, and Guesnerie 1992).

For linear models with predetermined variables, (Evans and Honkapohja 1994a) considered the stability under learning of sunspot solutions taking an autoregressive-moving average form, but they did not examine finite-state Markov solutions. Indeed, until the work of (Dvila 1997), it was not generally recognized that finite-state Markov sunspot solutions could exist in models with predetermined variables.[2] (Dvila 1997) and (Dvila and Guesnerie 2001) give conditions for existence of finite-state Markov solutions in both linear and nonlinear nonstochastic models with memory.

The current paper analyzes stability under learning of finite-state Markov equilibria in linear models with predetermined variables. We begin with nonstochastic models and the equilibria studied by Dávila. In the process of obtaining our results we uncover another class of solutions that has not

[1](Desgranges and Negroni 2001) have obtained conditions for eductive stability of two-state Markov stationary sunspot equilibria in an overlapping generations model.

[2]An exception is (Howitt and McAfee 1992). However this model relied on a nonlinear model that produced multiple steady states. (Evans, Honkapohja, and Romer 1998) also relied on finite-state Markov sunspot equilibria near distinct steady states.

previously been noted. These solutions, like the finite-state Markov solutions, depend on an extraneous k-state Markov process, but do not take on a finite number of values. We characterize these solutions and also analyze their stability under adaptive learning.

To analyze the stability under learning of these sunspot equilibria, we begin by showing that each of these rational expectations equilibria (REE) can be obtained as a solution to a member of a certain class of linear difference equations: we call these equations *representations* of the equilibria. Representations are most easily characterized as fixed points of a map T, which takes agents' perceived law of motion to the corresponding actual law of motion. This T-map, and hence the corresponding fixed points, will depend on the transition probabilities π of the associated Markov process.

It turns out that even if the model's parameters are such that k-SSEs exist, the corresponding transition probabilities π must satisfy certain constraints. On the other hand, whether or not π satisfies these conditions, a T-map is still defined, and to every non-trivial fixed point of this T-map corresponds at least one associated REE that depends explicitly on the Markov process. These observations lead us to the following definition. We call any solution to our representations *k-state dependant sunspot equilibria* and, anticipating their time-series properties, we define *ergodic k-SSEs* to be those k-state dependant sunspot equilibria which are not k-SSEs.

To obtain specific results about the existence of ergodic k-SSEs and to analyze the stability under learning of k-SSEs and ergodic k-SSEs, we consider in detail the case $k = 2$. We find that ergodic 2-SSEs exist whenever the model is indeterminate and the associated roots are real, whereas 2-SSEs exist in only part of this region. If agents use the functional form of these representations as their perceived law of motion, i.e. their regression model, then provided the parameters are appropriately restricted to be in certain proper subsets of the regions of existence, agents can learn the true form of the representation. Numerical simulations illustrate convergence under learning for both types of equilibria.

To keep close to the literature, and for presentational clarity, our results are initially presented for nonstochastic models. In applied work stochastic linear or linearized models are more typical. We show that all of our existence and stability results carry over to stochastic models with white noise shocks, in which the analogous sunspot solutions are noisy 2-SSEs and noisy ergodic 2-SSEs. In particular we show that for appropriate model parameter values these solutions can be stable under learning. Finally, we illustrate our results for a Cagan-type model and for Sargent's extension of the Lucas-Prescott model of investment under uncertainty to incorporate tax distortions and externalities.

2 The Model

We consider the nonstochastic model

$$y_t = \beta E_t y_{t+1} + \delta y_{t-1}. \tag{1}$$

We assume throughout that $\beta \neq 0$, $\delta \neq 0$ and $\beta + \delta \neq 0$. For simplicity of presentation we have omitted a constant intercept from the model. If instead, say, $y_t = \mu + \beta E_t y_{t+1} + \delta y_{t-1}$, then the model can be rewritten in the form (1) where y_t is reinterpreted as its deviation from $\hat{y} = \mu/(1-\beta-\delta)$. Throughout the paper we focus on doubly-infinite covariance-stationary processes.

(Dvila 1997) showed that finite-state Markov solutions to (1) could exist provided the Markov process is second-order. We begin by describing these solutions. For $n \leq k$, let S_n be the n^{th}-coordinate vector of unit length in $\mathbb{R}^k$, and let Δ_k be the set of all convex combinations of the vectors S_n. Notice that Δ_k is the $k-1$ unit simplex and thus elements of Δ_k represent probability distributions over the "states" S_n. Let $\mathcal{I}$ be the $k \times k$ lattice of positive integers, that is,

$$\mathcal{I} = \{1, \ldots, k\} \times \{1, \ldots, k\},$$

and set

$$P = \{\pi : \mathcal{I} \to \Delta_k\}.$$

Notice that P is simply the Cartesian product of Δ_k with itself k^2 times, and that an element of P may be identified with a $k \times k$ array of elements in Δ_k. A second order k-state Markov process (with states S_n) is a sequence of random variables s_t and a $k \times k$ array of probabilities $\pi \in P$ such that for all $i, j, n \in \{1, \cdots, k\}$,

$$prob\{s_{t+1} = S_n | s_{t-1} = S_i, s_t = S_j\} = \pi_{ij}(n).$$

We identify a k-state Markov process with its transition array π. A *k-state Markov stationary sunspot equilibrium* (k-SSE) is a pair $(\pi, \overline{y})$, where π

is a k-state second order Markov process and $\overline{y} \in \mathbb{R}^k$, with $\overline{y}_i \neq \overline{y}_j$ for $i \neq j$, is such that

$$y_t = \overline{y}_i \Leftrightarrow s_t = S_i \tag{2}$$

satisfies (1). We will also sometimes refer to y_t as a k-state Markov sunspot, since y_t itself follows a second-order k-state Markov process.

By explicitly considering the restrictions imposed by the model, we can obtain a set of linear equations, any solution to which yields a k-state Markov sunspot of the nonstochastic model. For each m and n write $\pi_{mn} \in \Delta_k$ as a column vector. If y_t satisfies (2) then

$$E_t y_{t+1} = \pi'_{mn}\overline{y} \Leftrightarrow s_{t-1} = S_m \text{ and } s_t = S_n.$$

We conclude that the pair $(\pi, \overline{y})$ is a k-SSE if and only if

$$\overline{y}_n - \delta\overline{y}_m = \beta\pi'_{mn}\overline{y} \quad \forall n, m \in \{1, \cdots, k\}. \tag{3}$$

(3) represents a homogeneous system of k^2 linear equations. Thus $\overline{y}_i = 0$ for all i is always a solution; this trivial sunspot coincides with the solution $y_t = 0$.

Existence of non-trivial solutions requires the system of equations to be dependant; this requirement imposes restrictions on the possible values of the parameters. Further restrictions are imposed by the requirements that the $\overline{y}_i$ be distinct (so that the k-state sunspot is not degenerate) and that the transition array represents legitimate probability distributions. (Dvila 1997) and (Dvila and Guesnerie 2001) demonstrated existence for a subset of the parameter space.

Proposition 1 *(Dávila). A k-SSE of (1) exists if and only if*

1. $-1 < \frac{1-\delta}{\beta} < 1$
2. $-1 < \frac{1+\delta}{\beta} < 1$

Given parameter values satisfying these conditions, the system (3) can be used to construct a k-SSE. Note further that since this system is homogeneous, the existence of one non-trivial solution implies the existence of a continuum of non-trivial solutions.[3] Finally, we remark that the region specified in Proposition 1 is a proper subset of the indeterminacy region. The latter is given by the union of the two regions (i) $\beta + \delta > 1$ with $|\delta| < |\beta|$ and (ii) $\beta + \delta < -1$ with $|\delta| < |\beta|$.

[3] The system (3) can be obtained from Dávila's paper by noting that the solution set to the ij^{th}-equation in (3) is equivalent to his manifold V_{ij}.

3 Representations

We now distinguish between a rational expectations equilibrium (REE) and its representation. An REE of the model is any stochastic process y_t which satisfies the associated expectational difference equation (1). A rational expectations equilibrium representation of the model is a linear difference equation, any solution to which is an REE. The importance of this distinction stems from the fact that the analysis of stability under econometric learning requires the specification of a representation; in fact, most accurately, it is the representation, not the REE, that is or is not stable under learning. Furthermore, the stability of a particular REE may depend on the associated representation induced by the perceived law of motion; for a detailed analysis of these topics for the model allowing also for intrinsic shocks, see (Evans and McGough 2005). In this section, we develop representations of k-SSEs that can be used to analyze their stability under learning.

Representations can be obtained as fixed points of a mapping called the T-map, and thus we begin with its construction, which will be greatly facilitated by the following lemma:[4]

Lemma 1 *Let A be a $k \times k$ matrix. Then there exists matrix $B = B(A)$, depending on π, such that*

$$E_t s_t' A s_{t+1} = s_{t-1}' B s_t.$$

To construct the T-map, we begin by specifying a perceived law of motion (PLM), that is, a functional form of the representation in terms of parameters (coefficients) and observables. A PLM may also be thought of as the forecasting model used by agents when forming expectations. These expectations may then be imposed into the reduced form model (1), which we now interpret as holding outside of an REE:

$$y_t = \beta E_t^* y_{t+1} + \delta y_{t-1}, \tag{1'}$$

where $E_t^* y_{t+1}$ denotes the forecast corresponding to the PLM. Equation (1'), with agents' forecasts imposed, determines the true data generating process, or actual law of motion (ALM). If the PLM is well-specified, then the ALM will have the same functional form as the PLM, thus inducing a map, called the T-map, from the perceived parameters of the forecasting model, to the actual parameters of the data generating process.

[4]Proofs of results presented in the main text are given in Appendix 1.

Using the above lemma as a guide, we take our PLM to be

$$y_t = ay_{t-1} + s'_{t-1}As_t. \tag{4}$$

For any real number x, denote by $I_k(x)$ the $k \times k$ matrix with x in each diagonal entry and zeros elsewhere. Now notice that

$$\begin{aligned} E_t^* y_{t+1} &= a^2 y_{t-1} + aE_t s'_{t-1}As_t + s'_{t-1}B(A)s_t \\ &= a^2 y_{t-1} + s'_{t-1}\left(I_k(a)A + B(A)\right)s_t. \end{aligned}$$

Inserting this into the reduced form equation (1) yields the actual law of motion (ALM) and thus determines the output of the T-map.[5] Set

$$T_1(a) = \beta a^2 + \delta \tag{5}$$

$$T_2(a, A) = I_k(\beta)\left(I_k(a)A + B(A)\right). \tag{6}$$

Then the ALM can be written

$$y_t = T_1(a)y_{t-1} + s'_{t-1}T_2(a, A)s_t.$$

Notice that a fixed point of the T-map determines a representation of an REE. We denote by $\Omega(\pi) \subset \mathbb{R} \times \mathbb{R}^{k \times k}$ the collection of fixed points of T; the index π reflects the fact that the T-map, and hence the set of fixed points, depends on the matrix of transition probabilities π. A fixed point of $T_1(a)$ is a real root a_i, for $i = 1, 2$, of the quadratic $\beta a^2 - a + \delta = 0$. The quadratic has real roots if and only if $\beta\delta \leq 1/4$. We have the following result.

Proposition 2 *Assume the parameters of the model are such that k-SSEs exist and the a_i are real. Let y_t be a stationary rational expectations equilibrium. If y_t is a k-SSE with associated transition array π, then there exists a point $(a, A) \in \Omega(\pi)$ such that $y_t = ay_{t-1} + s'_{t-1}As_t$.*

Note that $(a_i, 0) \in \Omega(\pi)$ so that $\Omega(\pi)$ is not empty. We say that $\Omega(\pi)$ is non-trivial if it contains points other than $(a_i, 0)$. The above result verifies that there are non-trivial fixed points to the T-map. Furthermore, it shows that any k-SSE can be represented as a fixed point of the T-map.

[5] Implicitly we are assuming that when expectations are formed the information set includes s_t, s_{t-1} and y_{t-1} but not y_t. See (Evans and McGough 2005) for further details and a discussion of the case in which y_t is also included in the information set.

In the following sections we will use the T-map to analyze stability under learning of k-SSEs using the E-stability principle. This will enable us to provide additional model parameter restrictions required for learning stability. However, the above proposition also raises an interesting question: Are there REE having representations of the form (4) that are not themselves k-SSEs? To address this question more formally we make the following definition: A *k-state dependant sunspot equilibrium* is any process y_t satisfying (4) for some transition array π and associated fixed point (a, A). Obviously, a k-SSE is a k-state dependant sunspot equilibrium; we define an *ergodic k-SSE* to be a k-state dependant sunspot equilibrium which is not a k-SSE. The following natural questions arise:

1. Do ergodic k-SSEs exist?
2. If so,
   (a) are they stable under learning?
   (b) if the transition array π corresponds to a k-SSE, then do ergodic k-SSEs exist with respect to this π?

The relevance of question 2, b, is the following: if no such ergodic k-SSE exist, then, when we show stability under learning, we can be confident that our agents are learning a k-SSE, and not an ergodic k-SSE. These questions appear difficult to address in the general case. For the case $k = 2$ we will find the answers to be "yes", "sometimes", and "no", respectively.

4 E-stability

Consider the PLM (4) and write $\theta = (a, A)$ and $T(\theta) = (T_1(a), T_2(a, A))$. Note that T maps $\mathbb{R} \times \mathbb{R}^{k\times k}$ into itself. Let θ^* be a fixed point of the T-map. We say θ^* (and the associated representation) is *E-stable* (or "expectationally stable") provided the differential equation

$$\frac{d\theta}{d\tau} = T(\theta) - \theta \tag{7}$$

is locally asymptotically stable at θ^*.[6] The *E-stability Principle* says that if the REE is E-stable then it is learnable by a reasonable adaptive algorithm.

[6] Here τ captures "notional" time: see (Evans and Honkapohja 2001) for details.

This principle is known to be valid for least squares and closely related statistical learning rules in a wide variety of models. For a thorough discussion see (Evans and Honkapohja 2001).[7] Let $DT(\theta^*)$ be the derivative of the T-map evaluated at θ^*. In the typical case of a locally unique fixed point θ^*, and making the standard regularity assumption that no eigenvalue of $DT(\theta^*)$ has a real part equal to one, necessary and sufficient conditions for E-stability are that all eigenvalues of $DT(\theta^*) - I$, the derivative of $T(\theta) - \theta$ evaluated at θ^*, have real parts less than zero.

However, the definition of expectational stability just given is inadequate when there is a non-trivial connected set of rest points of the differential equation (7), as is the case for our model: if $\Omega(\pi)$ is locally connected then no point in Ω is locally asymptotically stable. In this context we restate the notion of E-stability as follows: we say that a set of fixed points, Q, is E-stable provided there is a neighborhood U of Q such that for any $\theta_0 \in U$ the trajectory of θ determined by the differential system (7) converges to a point in Q. A necessary condition for E-stability of Q is that for all $q \in Q$, the non-zero eigenvalues of the derivative of $T(\theta) - \theta$ evaluated at q have negative real part. Sufficient conditions are in general difficult to obtain, because of the presence of zero eigenvalues, but in the case of a single zero eigenvalue these necessary conditions are sufficient. This is proved in Proposition 11 in Appendix 2.

Assume the parameters of the model are such that sunspots exist and the roots of the associated quadratic, $\beta a^2 - a + \delta = 0$, are real. To analyze the stability of $\Omega(\pi)$ in this case, begin by noticing that these real roots are the fixed points of T_1 and are given by

$$a_1 = \frac{1 - \sqrt{1 - 4\beta\delta}}{2\beta} \quad \text{and} \quad a_2 = \frac{1 + \sqrt{1 - 4\beta\delta}}{2\beta}.$$

It follows that $\Omega(\pi) = \Omega_1(\pi) \cup \Omega_2(\pi)$ where

$$\Omega_i(\pi) = \{(a_i, A) \in \Omega(\pi)\}.$$

Since T_1 is decoupled from T_2, we can analyze its stability independently. We have that $DT_1(a) = 2\beta a$, which immediately implies that the subsystem

[7] The connection between statistical learning and E-stability is established using convergence results from the stochastic approximation literature. This technique is described in (Marcet and Sargent 1989), (Woodford 1990) and Chapters 6 and 7 of (Evans and Honkapohja 2001).

in T_1 is locally asymptotically stable if and only if $a = a_1$. Thus we have the following proposition:

Proposition 3 *The set* $\Omega_2(\pi)$ *is not E-stable.*

Note that $DT_1(a_1) < 1$, which gives us hope that $\Omega_1(\pi)$ may be E-stable. It suffices to show that $(a_1, A) \in \Omega_1(\pi)$ implies that the eigenvalues of $DT_2(a_1, A)$ have real part less than one. In general this appears to be difficult to demonstrate. In the following Section we analyze the case $k = 2$.

5 2-State Sunspots

To obtain explicit results concerning the existence and representations of ergodic k-SSEs, as well as to derive specific stability results for both k-SSEs and ergodic k-SSEs, we now use the theory set out above to consider in detail the case $k = 2$. In the process we obtain a simple method by which both 2-SSEs and ergodic 2-SSEs can be constructed.

5.1 Existence of 2-SSEs

Many of the details presented in this subsection are contained in (Dvila and Guesnerie 2001); we include them here for completeness. The pair $(\pi, \overline{y})$ is a 2-SSE for the model (1) if and only if

$$\begin{aligned} \pi_{11}(1) + \pi_{22}(2) &= 1 + \beta^{-1}(1 - \delta), &(8)\\ \pi_{21}(1) + \pi_{12}(2) &= 1 + \beta^{-1}(1 + \delta), &(9)\\ \pi_{22}(1)\overline{y}_1 + \pi_{11}(2)\overline{y}_2 &= 0, &(10)\\ (1 + \delta\pi_{21}(1) - \pi_{12}(2))\overline{y}_1 &= (\pi_{21}(1) - (1 + \delta\pi_{12}(2)))\overline{y}_2. &(11) \end{aligned}$$

For the sunspot to be non-trivial, it must be that $\overline{y}_1 \neq \overline{y}_2$, which, by restrictions (10) and (11), implies

$$\frac{\pi_{22}(1)}{1 + \delta\pi_{21}(1) - \pi_{12}(2)} = \frac{\pi_{11}(2)}{1 + \delta\pi_{12}(2) - \pi_{21}(1)}. \tag{12}$$

Restrictions (8),(9), and (12) can be combined to determine the transition array up to one degree of freedom. Equation (10) or (11) can then be used to determine the ratio of the states.

According to the preceding arguments, 2-SSEs exist provided transition arrays satisfying (8),(9), and (12) exist. The restrictions on the transition probabilities can be rewritten as

$$\pi_{11} = \frac{1-\delta}{\beta} + \pi_{22},\ \pi_{12} = -\frac{\delta}{\beta} + \pi_{22} \text{ and } \pi_{21} = \frac{1}{\beta} + \pi_{22}, \tag{13}$$

where, for notational simplicity, we write $\pi_{ij} = \pi_{ij}(1)$. Set

$$\begin{aligned} L(\beta,\delta) &= \max\{\frac{\delta-1}{\beta}, \frac{\delta}{\beta}, -\frac{1}{\beta}\}, \\ U(\beta,\delta) &= \min\{1-\frac{1-\delta}{\beta}, 1+\frac{\delta}{\beta}, 1-\frac{1}{\beta}\}. \end{aligned}$$

Imposing $\pi_{ij} \in (0,1)$ yields the following: A 2-SSE exists if and only if

$$(0,1) \cap (L(\beta,\delta), U(\beta,\delta)) \neq \emptyset,$$

where we say $(L,U) = \emptyset$ if $U \leq L$. A straightforward argument then shows that this set is non-empty if and only if β and δ satisfy the restrictions in Proposition 1.

FIGURE 1 ABOUT HERE

In Figure 1 the regions of parameters corresponding to existence of 2-SSEs and, simultaneously, real roots of the quadratic, are denoted by A_1 and B_1. Regions A_2 and B_2 denote those parts of the indeterminacy regions in which there are real roots but 2-SSEs do not exist.

5.2 Representations

The T-map can be explicitly computed as

$$T_1(a) = \beta a^2 + \delta,$$

$$T_2(a,A) = \begin{bmatrix} \beta(aA_{11} + \pi_{11}(1)A_{11} + \pi_{11}(2)A_{12}) & \beta(aA_{12} + \pi_{12}(1)A_{21} + \pi_{12}(2)A_{22}) \\ \beta(aA_{21} + \pi_{21}(1)A_{11} + \pi_{21}(2)A_{12}) & \beta(aA_{22} + \pi_{22}(1)A_{21} + \pi_{22}(2)A_{22}) \end{bmatrix}.$$

The fixed points of T_1 are $a = \frac{1 \pm \sqrt{1-4\beta\delta}}{2\beta}$, and the fixed points of T_2 are determined by the following four equations:

$$\begin{aligned}
(1/\beta - a)A_{11} &= \pi_{11}A_{11} + (1-\pi_{11})A_{12} \\
(1/\beta - a)A_{12} &= \pi_{12}A_{21} + (1-\pi_{12})A_{22} \\
(1/\beta - a)A_{21} &= \pi_{21}A_{11} + (1-\pi_{21})A_{12} \\
(1/\beta - a)A_{22} &= \pi_{22}A_{21} + (1-\pi_{22})A_{22}.
\end{aligned}$$

This linear homogeneous system has non-trivial solutions only if linear dependence is exhibited.

To investigate this, notice that we can write $A_{ij} = K_{ij}A_{12}$ where

$$\begin{aligned}
K_{11} &= \frac{1-\pi_{11}}{1/\beta - a - \pi_{11}}, \\
K_{21} &= \frac{\pi_{21}K_{11} + 1 - \pi_{21}}{1/\beta - a}, \\
K_{22} &= \frac{\pi_{22}K_{21}}{1/\beta - a - (1-\pi_{22})}.
\end{aligned}$$

We conclude that a non-trivial solution exists if and only if the following equation holds:

$$1/\beta - a = \pi_{12}K_{21} + (1-\pi_{12})K_{22}. \tag{14}$$

Recall it was shown that 2-state sunspots exist if and only if the transition array satisfies (13). Algebra (Mathematica) shows that imposing these restrictions on π implies that equation (14) holds. Notice that, in this case, there exists a one-dimensional continuum of representations: the choice of A_{12} is free, but once made, the remaining A_{ij} are pinned down. Further, to construct a 2-SSE, simply choose $\pi_{22} \in (L(\beta,\delta), U(\beta,\delta))$ and pick A_{12} arbitrarily. The above equations can then be used to determine the remaining parameter values.

5.3 Existence of ergodic 2-SSEs

In Section 3 we wondered, in question 2b, for π satisfying (13), whether fixed points determining representations of ergodic 2-SSEs existed.

Proposition 4 *Suppose β and δ are such that 2-SSEs exist and the roots of the associated quadratic are real. Suppose a denotes a root of this quadratic. Let π satisfy the existence restrictions (13). If $T_2(a, A) = A$ and*

$$y_t = ay_{t-1} + s'_{t-1}As_t,$$

then y_t is a 2-SSE.

Proposition 4 implies that if π satisfies (13), and if $\Omega_1(\pi)$ is E-stable, then agents will necessarily be learning a 2-SSE, and not possibly an ergodic 2-SSE.

Next we have the following result concerning the existence of ergodic 2-SSEs.

Proposition 5 *If (β, δ) is in the indeterminate region and the roots of the usual quadratic are real then there exist transition arrays π such that $\Omega(\pi)$ is non-trivial. Furthermore, π can be chosen to violate (13), i.e. ergodic 2-SSEs exist.*

The result implies that provided the model is indeterminate, there exist ergodic 2-SSEs. Thus, in Figure 1, ergodic 2-SSEs exist throughout regions A_1, A_2, B_1 and B_2. We conjecture that the above results hold for $k > 2$. Also notice that while the transition array π need not satisfy (13) for 2-state dependent sunspot equilibria to exist, and, in fact, when $(\beta, \delta) \in A_1 \cup B_1$, cannot satisfy (13) for ergodic 2-SSEs to exist, there may still be restrictions on its values. In fact we have the following result on the dimensionality of ergodic 2-SSEs.

Proposition 6 *Let I be the unit cube in $\mathbb{R}^4$. Let $R = \{\pi \in I$ such that (13) holds$\}$ and $\hat{R} = \{\pi \in I$ such that $\Omega(\pi)$ is nontrivial$\}$. Then (i) R is homeomorphic to $\mathbb{R}$ and (ii) there is a subset of $\hat{R}$ that is homeomorphic to $\mathbb{R}^3$, but $\hat{R}$ has Lesbesgue measure zero as a subset of I.*

Notice that $R \subset \hat{R}$, and for $\pi \in R$ there exist 2-SSEs while if $\pi \in \hat{R} \setminus R$ there exist ergodic 2-SSEs. Thus this proposition shows that the set of transition arrays for which ergodic 2-SSEs exist is much bigger than the set for which 2-SSEs exist. However, $\hat{R}$ is still very restrictive since it has measure zero as a subset of I.

We can also characterize the range of ergodic 2-SSEs. We have

Proposition 7 *Given a and A, assume that y_t is an ergodic 2-SSE. Let $\mathcal{R} \subset \mathbb{R}$ be the range of y_t. Then $\mathcal{R}$ is uncountable and if $F \subset \mathcal{R}$ is finite then prob$\{y_t \in F\} < 1$.*

Thus ergodic 2-SSEs are qualitatively very different from 2-SSEs. Both types of solution are driven by second-order 2-state Markov processes, but ergodic 2-SSEs can take infinite many values, while 2-SSEs take only two values. The distinction between 2-SSEs and ergodic 2-SSEs does not appear in the purely forward looking model and arises specifically because of the dependence of y_t on its previous value.

The complex behavior of ergodic 2-SSEs is revealed through simulations. Figures 2.1 and 2.2 give the time paths for y_t of an ergodic 2-SSE for two different choices of (β, δ), as indicated in the Figure. The time series of these ergodic 2-SSEs are particularly intriguing because they show complicated dynamics even though there is no intrinsic noise and the process is driven entirely by an exogenous two state Markov process. This shows the clear potential of stationary sunspot equilibria, driven by finite-state Markov processes, for explaining complex economic fluctuations.

Figure 2 Here

To gain intuition for the behavior witnessed in Figure 2, we may further analyze the dynamic properties of ergodic k-SSEs via the phase space. Recall that ergodic k-SSEs have representations of the form

$$y_t = a_1 y_{t-1} + s'_{t-1} A s_t. \tag{15}$$

In (Evans and McGough, 2005), we showed that every rational expectations equilibrium of the reduced form model (1) has a common factor representation; this means that there exists a martingale difference sequence ε_t so that,

$$y_t = a_1 y_{t-1} + \hat{\eta}_t, \tag{16}$$

$$\hat{\eta}_t = a_2 \hat{\eta}_{t-1} + \varepsilon_t. \tag{17}$$

We must conclude that $\hat{\eta}_t = s'_{t-1} A s_t$, so that $s'_{t-1} A s_t$ is a first order markov process. This equivalence also tells us that $\hat{\eta}_t$ has k^2 states. If $k = 2$, then we can compute that the possible states for $\hat{\eta}_t$ are $\eta \equiv (\eta_1, \ldots, \eta_4)' = vec(A')$, and the associated transition matrix P is

$$P = \begin{pmatrix} \pi_{11}(1) & \pi_{11}(2) & 0 & 0 \\ 0 & 0 & \pi_{12}(1) & \pi_{12}(2) \\ \pi_{21}(1) & \pi_{21}(2) & 0 & 0 \\ 0 & 0 & \pi_{22}(1) & \pi_{22}(2) \end{pmatrix}.$$

Equation (16), together with the fact that $\hat{\eta}_t$ is a four state process, suggests that we may think of the dynamics of y_t as being determined by state-contingent linear systems. In particular, as long as, say, $\hat{\eta}_t = \eta_i$, y_t will converge toward $(1-a_1)^{-1}\eta_i$, which is the steady-state associated to the dynamic system $y_t = a_1 y_{t-1} + \eta_i$. This convergence will continue until $\hat{\eta}_t$ switches states, to, say, η_j, and the system begins converging toward $(1 - a_1)^{-1}\eta_j$. These conditionally linear dynamics are further restricted by the transition matrix P, which imposes that η_2 and η_3 are reflecting states, i.e. $\hat{\eta}_t = \eta_2 \Rightarrow \hat{\eta}_{t+1} \neq \eta_2$, etc.

To gain further intuition about the dynamics of ergodic 2-SSEs, consider Figure 3 which describes a case when $k = 2$. Here, in phase space, i.e. in the (y_{t-1}, y_t)-plane, are plotted the four lines, labeled L_i, which dictate the dynamics of y_t in each of the four states: note that the slope of each line is a_1, so that the lines differ only in the intercept terms, which correspond to the η_i's. The 45° line is plotted as well. Set

$$\bar{y}_1 = (1-a_1)^{-1}\eta_1 \quad \text{and} \quad \bar{y}_2 = (1-a_1)^{-1}\eta_4.$$

Now consider the dynamics implied by the initial conditions $y_0 = \bar{y}_2$ and $\hat{\eta}_0 = \eta_4$. If $\hat{\eta}_1 = \eta_4$ then $y_1 = y_0$. If $\hat{\eta}_1 = \eta_3$ then follow the arrows first to L_3, and then to the point labeled "B." The horizontal (and vertical) position of this point corresponds to y_1. If $\hat{\eta}_t$ remained in state η_3, then y_t would converge toward point "C"; however, according to the transition matrix P, that never happens. The value of $\hat{\eta}_2$ will either be η_1, in which case we follow the solid arrows, or it will be η_2, in which case we follow the dotted arrows; the process continues with the relevant dynamic system being determined in each time t by the value of $\hat{\eta}_t$. Figure 4, which plots a simulation of an ergodic 2-SSE, supports the intuition provided by this diagram.

Figures 3 and 4 Here

Now consider the case in which the 2-state dependant sunspot equilibrium corresponds to a 2-SSE. Then (16) implies that the two possible states for y_t are given by $\bar{y}$ as defined above, and further, the states η are restricted by the conditions

$$\eta_3 = (1 - a_1)^{-1}(\eta_1 - \eta_4) = -\eta_2.$$

Graphically, this corresponds to the point B coinciding with $\bar{y}_1$: see Figure 5.

Figure 5 Here

One more result may be obtained using the representation (16). Though we have throughout assumed that the equilibrium processes under consideration are doubly infinite, many applications require the specification of an initial condition. The next result shows that in case the transition matrix corresponds to a 2-SSE, even if the initial condition does not meet the restrictions imposed by (10) and (11), convergence to a 2-SSE still obtains.

Proposition 8 *Let $\hat{\eta}_t$ be the common factor sunspot associated to the 2-SSE y_t. Let $\hat{y}_0 \in \mathbb{R}$ and $\hat{y}_t = a\hat{y}_{t-1} + \hat{\eta}_t$. Then $|\hat{y}_t - y_t| \to 0$ almost surely.*

To prove this, simply notice that

$$\hat{y}_t = a_1\hat{y}_0 + \sum_{k=0}^{t-1} a_1^k \hat{\eta}_{t-k},$$

and similarly for y_t. Thus $|\hat{y}_t - y_t| = |a_1|^t |y_0 - \hat{y}_0| \to 0$.

5.4 E-stability of 2-SSEs and ergodic 2-SSE s

We can now use the explicit form of the T-map to analyze E-stability. As observed in Section 4, to show that $\Omega_1(\pi)$ satisfies the necessary condition for E-stability, it suffices to analyze the subsystem T_2 for $a = a_1$. We have that

$$\frac{\partial vec(T_2)}{\partial vec(A)} = \begin{bmatrix} \beta(a_1 + \pi_{11}) & \beta(1 - \pi_{11}) & 0 & 0 \\ 0 & \beta a_1 & \beta\pi_{12} & \beta(1 - \pi_{12}) \\ \beta\pi_{21} & \beta(1 - \pi_{21}) & \beta a_1 & 0 \\ 0 & 0 & \beta\pi_{22} & \beta(a_1 + 1 - \pi_{22}) \end{bmatrix}. \quad (18)$$

E-stability requires that the real parts of the eigenvalues of this derivative must be less than one when evaluated at points in Ω_1. We analyze the stability of 2-SSEs and ergodic 2-SSEs separately.

First, notice that the above derivative is independent of the matrix A. Also, provided the transition array π satisfies the restrictions (13), it can be verified algebraically that the eigenvalues are independent of the chosen transition array. Thus, analysis of the eigenvalues of DT_2 requires only varying β and δ. We further restrict the parameter space to guarantee existence of representable 2-SSEs. Specifically, we assume that parameters satisfy the conditions in Proposition 1, and further that $\beta\delta < 1/4$, so that the associated roots are real. Finally, we consider negative and positive values of β separately, labeling the relevant regions "Area A_1" and "Area B_1" respectively: see Figure 1. We have

Proposition 9 *Consider 2-SSEs. (1) The set Ω_1 is E-stable for parameters in Area A_1. (2) The set Ω_1 is unstable for parameters in Area B_1.*

To assess the E-stability of ergodic 2-SSEs, we again analyze the eigenvalues of (18). However, when $(\beta, \delta) \in A_1 \cup B_1$, the transition arrays necessarily do not satisfy (13), and, in fact, these eigenvalues may depend on the values of the probabilities π. This has the unfortunate consequence of increasing the dimension of the parameter space and making analytic results intractable. Specifically, we must now consider stability of ergodic 2-SSEs for different β, δ, and π. To establish existence of stable ergodic 2-SSEs, we proceed as follows: for values of (β, δ) in each of the four regions A_i, B_i, $(i = 1, 2)$, we use the method described in the Appendix to choose a value of π to which corresponds ergodic 2-SSEs, and then we analyze the stability of the associated set of fixed points $\Omega_1(\pi)$ by numerically computing the eigenvalues of (18). We obtain the following result.

Proposition 10 *Consider ergodic 2-SSEs. (1) For $i = 1, 2$, there exist $(\beta, \delta) \in A_i$ and transition array π violating (13) such that $\Omega_1(\pi)$ is non-trivial and is E-stable. (2) For $i = 1, 2$, there exist $(\beta, \delta) \in B_i$ and transition array π violating (13) such that $\Omega_1(\pi)$ is not E-stable.*

This result implies that there exist stable ergodic 2-SSEs for values (β, δ) such that 2-SSEs do not even exist. Our numerical are stronger than the above Proposition indicates: the stability of $\Omega_1(\pi)$ obtained for all values of

$(\beta, \delta) \in A_i$ and for all transition arrays π that we tested. Similarly for all values of $(\beta, \delta) \in B_i$ and π tested, we found instability. Based on this we make the following conjecture.

Conjecture. *(1) If $(\beta, \delta) \in A = A_1 \cup A_2$ and if π is such that $\Omega_1(\pi)$ is non-trivial then $\Omega_1(\pi)$ is E-stable. (2) If $(\beta, \delta) \in B = B_1 \cup B_2$ and if π is such that $\Omega_1(\pi)$ is non-trivial, then $\Omega_1(\pi)$ is not E-stable.*

According to the conjecture, for 2-state dependant sunspot equilibrium, the E-stable region is given by $A = A_1 \cup A_2$. Summarizing, in region A_1 we know that 2-SSEs exist and are E-stable, while in region A_2 2-SSEs do not exist. E-stable ergodic 2-SSEs exist in A and ergodic 2-SSEs are conjectured to be E-stable throughout A.

6 Simulations

In this Section we provide the standard real-time learning algorithm based on recursive least-squares and present numerical simulations illustrating the stability of 2-SSEs and ergodic 2-SSEs.

Agents are assumed to have the PLM (4), reproduced here for convenience,

$$y_t = ay_{t-1} + s'_{t-1}As_t,$$

and are assumed to use OLS (ordinary least squares) to estimate the associated parameters a and A. Set

$$X_t = [y_{t-1}, s_{t-1}(1)s_t(1), s_{t-1}(1)s_t(2), s_{t-1}(2)s_t(1), s_{t-1}(2)s_t(2)]',$$

where $s_t(i)$, for $i = 1, 2$, denotes the components of s_t, and write $\theta = [a, \text{vec}(A)']'$, where $\text{vec}(A)$ is the operator that stacks in order the columns of A into a column vector. (Earlier we defined $\theta = (a, A)$ but it is now convenient to rewrite θ as a column vector). This allows us to write the stochastic process for the estimators recursively[8] as

$$\begin{aligned} \theta_t &= \theta_{t-1} + \frac{1}{t}R_t^{-1}X_t(y_t - \theta'_{t-1}X_t) \\ R_t &= R_{t-1} + \frac{1}{t}(X_tX'_t - R_{t-1}) \end{aligned}$$

[8]See, e.g., (Marcet and Sargent 1989) or pp. 32-3 of (Evans and Honkapohja 2001).

where

$$y_t = T_1(a_{t-1})y_{t-1} + s'_{t-1}T_2(a_{t-1}, A_{t-1})s_t.$$

The term t^{-1} in the recursive algorithm is called the "gain sequence." This or closely related gain sequences arise in least squares and other statistical estimators. The behavior of this algorithm was analyzed via simulations. The algorithm was initialized by choosing points at random within a given neighborhood of the set $\Omega_1(\pi)$.

Analytic results implying convergence with probability one typically require amending the algorithm with a projection facility. Alternatively, one can adjust the gain of the algorithm to obtain convergence with probability approaching one. For the simulations produced here we scale the gain of the RLS algorithm by 1/25, thus increasing the probability of convergence.

Our results on E-stability show that convergence to a 2-SSE obtain only if the model's parameter values are chosen to lie in Area A_1. We chose several different parameter pairs in this area and for each pair ran several simulations. For each pair of values we found that with positive probability, i.e. for a positive proportion of the simulations, convergence to $\Omega_1(\pi)$ appears to obtain.[9] Similar results obtain when ergodic 2-SSEs were analyzed in region A.

FIGURE 6 HERE

Figure 6 shows portions of the time series for y_t for a simulation in which there is convergence to a 2-SSE. In this figure we set $\beta = -3$ and $\delta = 1$. Here REE denotes the path under fully rational expectations, while RTL denotes the path (for the same sequence of random sunspot shocks) under real time learning.[10] The final segments, shown in the northeast and southeast panels, are for a time period after learning has nearly converged.

7 Extension to Stochastic Models

In order to reduce the burden of the theoretical argument and to keep closer to the literature of finite-state sunspot equilibria, our results have been pre-

[9]For all simulations in which convergence did not appear to obtain, the norms of the estimates appear to diverge to infinity.

[10]The REE path illustrated in the Figures is obtained using the parameter vector $\theta \in \Omega_1(\pi)$ that is closest, in the Euclidean metric, to the terminal simulation value for θ_t under RTL.

sented in terms of the nonstochastic model (1). In applied work, however, stochastic models are often employed. In this Section we show how our results can be extended to models with intrinsic random shocks. To keep matters simple we consider the model:

$$y_t = \beta E_t y_{t+1} + \delta y_{t-1} + v_t, \tag{19}$$

where v_t is a white noise exogenous process and $E_t y_{t+1}$ denotes the mathematical expectation of y_{t+1} conditional on information available at t.

A rational expectations equilibrium of the stochastic model is any process y_t which solves (19). Again we restrict attention to doubly-infinite, covariance-stationary solutions. For a complete characterization of such solutions see (Evans and McGough 2005); there it is also shown that the norm of a doubly infinite, covariance stationary solution will be uniformly bounded in both conditional and unconditional expectation.

Let z_t be a stationary solution to the stochastic model (19) that does not depend on extrinsic noise: we will refer to z_t as a "fundamentals solution". It is well known that there exist such solutions to the noisy model (19) if and only if the associated quadratic $\beta a^2 - a + \delta$ has at least one root with norm less than one. If the model is determinate, then z_t is the unique REE, and if the model is indeterminate and if the roots of the associated quadratic are real, there are two fundamentals solutions. In either case, we may take the fundamentals solution z_t to have a representation of the form

$$z_t = a z_{t-1} + b v_t, \tag{20}$$

where a is a root of the associated quadratic and $b = (1 - \beta a)^{-1}$. Because of their parsimonious representation, these REE are often called minimal state variable (MSV) solutions.

We now couple an MSV solution with a k-state dependent sunspot equilibrium. Let ζ_t be a solution to the nonstochastic model (1) and let $y_t \equiv z_t + \zeta_t$. Since $E_t y_{t+1} = E_t z_{t+1} + E_t \zeta_{t+1}$ and $y_{t-1} = z_{t-1} + \zeta_{t-1}$, it is immediate that y_t is a solution to (19). If ζ_t is a k-state dependent sunspot equilibrium, we call y_t a noisy k-state dependent sunspot equilibrium; if ζ_t is a k-SSE, we call y_t a noisy k-SSE; and if ζ_t is an ergodic k-SSE, we call y_t a noisy ergodic k-SSE.

Woodford's conjecture, that the model must be indeterminate for stationary sunspot equilibria to exist, holds in both the stochastic and nonstochastic model, and indeterminacy in either model obtains when both roots of the above quadratic lie inside the unit circle; see e.g. (Evans and McGough 2005) for details. Thus if the model's parameters are such that noisy k-state dependent sunspot equilibria exist in the nonstochastic model then a stationary fundamentals solution of the noisy model exists. Combining this observation with the definition of a noisy k-state dependent sunspot equilibrium, we find that noisy k-state dependent sunspot equilibria and noisy ergotic 2-SEEs exist whenever the model is indeterminate, and when the restrictions on the model's parameters obtained by Dávila are met, noisy 2-SSEs exist. Although the result is almost immediate, it might appear surprising that the white noise disturbance, which may make any value of y_{t-1} possible, does not disturb the careful balance required for k-SSEs.

To analyze stability under learning, we specify the stochastic PLM

$$y_t = ay_{t-1} + s'_{t-1}As_t + bv_t. \tag{21}$$

The corresponding ALM is

$$y_t = T_1(a)y_{t-1} + s'_{t-1}T_2(a, A)s_t + T_3(a, b)v_t, \text{ where}$$

$$T_3(a, b) = \beta ab + 1.$$

The collection of fixed points is now $\tilde{\Omega}(\pi) \subset \mathbb{R} \times \mathbb{R}^{k\times k} \times \mathbb{R}$.

For the model (19) the analog of Proposition 2 becomes:

Proposition 2' *Assume the parameters of the model are such that noisy k-SSEs exist and the a_i are real. Let y_t be a stationary rational expectations equilibrium. If $y_t = z_t^i + \zeta_t$ is a noisy k-SSE with associated transition array π, then there exists a point $(a, A, b) \in \tilde{\Omega}(\pi)$ such that $y_t = ay_{t-1} + s'_{t-1}As_t + bv_t$.*

The E-stability results in Section 5.4 apply also to the case of the stochastic model (19). The stochastic case adds one additional E-stability condition from $T_3(a, b)$, namely $a\beta < 1$. Adding this condition clearly leaves the instability results unaffected. The stability results of Proposition 9 are unaffected because the stability requirements for 2-SSEs in the nonstochastic model already include the stronger requirement $2a\beta < 1$. Finally the results for Proposition 10 are numerical and analogous numerical results hold for noisy ergodic 2-SSEs. Thus all of the E-stability results from Section 5.4 carry over to the noisy versions of the equilibria in the model (19) with white noise shocks.

The algorithms for real-time recursive least-squares learning also extend to the stochastic model by expanding the regressors X_t to include v_t and now writing $\theta = [a, \text{vec}(A)', b]$. Under real-time learning the estimates the stochastic process followed by y_t is now given by

$$y_t = T_1(a_{t-1})y_{t-1} + s'_{t-1}T_2(a_{t-1}, A_{t-1})s_t + T_3(a_{t-1}, b_{t-1})v_t,$$

where under learning v_t is now also in the information set. Time paths of stable noisy 2-SSEs and stable noisy ergodic 2-SSEs show the kind of random irregular fluctuations typical in macroeconomic data, even though the sunspot process itself takes only two states.

8 Applications

In this section we consider two examples that illustrate the application of the theory developed above. We first briefly consider a Cagan-type model, which can be indeterminate, yet in which neither k-SSEs nor stable ergodic 2-SSEs exist. We then consider the (Sargent 1987) extension of the (Lucas and Prescott 1971) model of investment under uncertainty to allow for taxes and externalities. For this model there are parameter values that yield both stable 2-SSEs and stable ergodic 2-SSEs.

The discrete-time form of the Cagan model can be given as

$$p_t = \beta E_t p_{t+1} + \alpha m_t, \tag{22}$$

where β lies in the unit interval. Assume a money supply rule of the form

$$m_t = \overline{m} + \xi p_{t-1}. \tag{23}$$

A white noise shock to the money supply could easily be added. Combining equations yields the reduced form

$$p_t = \alpha\overline{m} + \beta E_t p_{t+1} + \alpha\xi p_{t-1}. \tag{24}$$

Let $\delta = \alpha\xi$. Since $\beta > 0$, this model is indeterminate provided $\delta > 1 - \beta$ and $\delta < \beta$. With the value of δ unrestricted, it follows that for $\frac{1}{2} < \beta < 1$ and appropriate δ, the model is indeterminate. However, with $0 < \beta < 1$ only the B_2 region of indeterminacy is feasible. Thus in this model k-SSEs do not exist. Ergodic 2-SSEs do exist, but will not be stable under learning.[11] This

[11] If the Cagan model is interpreted as obtained from a linearized overlapping generations model of money then $\beta < 0$ is possible. We do not pursue these cases here.

example shows that the requirement that sunspot equilibria be stable under learning can be a demanding test.

Our second example is based on Sargent's extension of the (Lucas and Prescott 1971) model of investment under uncertainty to allow for dynamic market distortions due to taxes and externalities; see Ch. XIV of (Sargent 1987).[12] Consider a competitive industry with N identical firms. Output x_t of the representative firm at t is given by

$$x_t = x_0 + f_0 k_t + f_1 K_t + f_2 K_{t-1},$$

where k_t is the capital stock of the individual firm and $K_t = N k_t$ denotes the aggregate capital stock. The presence of the two terms in K_t reflect contemporaneous and lagged external effects. These may be positive or negative, so we do not restrict the signs of f_1 or f_2, but we assume $f_0 > 0$ and $x_0 > 0$. Taxes are levied on firms on capital in place. The rate itself is assumed to depend on current and lagged aggregate capital stock, so that $\tau_t = g_0 + g_1 K_t + g_2 K_{t-1}$. Total output is $X_t = N x_t$, and market demand is

$$p_t = D - A X_t + u_t,$$

where u_t is white noise. We require $p_t \geq 0$. In this example we have included stochastic shocks, as is standard in this application, but we remark that nonstochastic versions are also sometimes considered, e.g. see Chapter IX of (Sargent 1987). The firm chooses k_t to maximize

$$E_0 \sum_{t=0}^{\infty} B^t \{ p_t (x_0 + f_0 k_t + f_1 K_t + f_2 K_{t-1}) - w k_t - \tau_t k_t - \frac{C}{2}(k_t - k_{t-1})^2 \},$$

where k_{-1} is given and w, the rental on capital goods, is for convenience assumed to be constant. $C > 0$ reflects adjustment costs for changing k_t. The Euler equation for this problem can be written

$$p_t f_0 - (w + \tau_t) + BCE_t^* k_{t+1} - C(1+B)k_t + C k_{t-1} = 0 \tag{25}$$

for $t \geq 0$. For an optimum solution for the firm we also require that $k_t \geq 0, x_t \geq 0$, and that the transversality condition is met.

[12] For further details of the temporary equilibrium set-up see (Evans and McGough 2005). Stability under learning of the MSV solutions was examined in Section 8.6.2 of (Evans and Honkapohja 2001).

In order to define the temporary equilibrium and study learning, we need to carefully specify the information structure. We assume that firms use observations of lagged capital stock, the current intrinsic exogenous shocks u_t and the current extrinsic exogenous variable s_t to make forecasts $E_t^* k_{t+1}$. Given these forecasts, firms choose their demands for capital k_t, conditional on p_t and τ_t, to satisfy (25). The temporary equilibrium is then given by the market clearing values of p_t, τ_t and k_t. Using the identical agent assumption, and combining equations, we obtain the reduced form

$$k_t = \mu + \beta E_t^* k_{t+1} + \delta k_{t-1} + \gamma u_t,$$

where $\beta = BC\Omega^{-1}$, $\delta = -(f_0 A f_2 N^2 + g_2 N - C)\Omega^{-1}$, $\Omega = f_0 A N(f_0 + f_1 N) + g_1 N + C(1 + B)$ and $\gamma = \Omega^{-1}$.

This form differs from our stochastic model only by the presence of a constant term. Incorporating the constant term into the above theory is straightforward and the details are left to the reader. The only issue concerns E-stability: the PLM must be modified to include a constant thus creating an addition component of the T-map. This component is not coupled with the system T_2 and analysis of its derivative is elementary. It can be shown that if the model without the constant exhibits stable k-state dependant sunspot equilibria, then the model with the intercept does as well.

With no externalities or taxes there is a unique stationary REE. However, in general the parameters β and δ are unrestricted. In particular, for some parameter regions the associated quadratic has both real roots inside the unit circle, and there are multiple stationary solutions, including stable noisy k-state Markov sunspots. This model is easy to study numerically. As already noted, when externality or tax distortions are present the indeterminacy case is possible. Furthermore, both 2-SSEs and ergodic 2-SSEs that are stable under learning arise in some regions of the parameter space. For example, normalizing with $N = 1$, the parameter values $A = 1, B = 0.95, C = 0.46, g_1 = -1, g_2 = 0.3, f_0 = 1, f_1 = -1, f_2 = 0.3$ leads to $\beta = -4.24$ and $\delta = 1.36$, which is in region A_1. Empirical work investigating k-state dependent sunspot equilibria in this model would therefore appear to be of considerable interest.

9 Conclusion

Finite state Markov stationary sunspot equilibria, in forward-looking models, played a central role in the early literature on expectations driven fluctuations. More recently they have received renewed interest because of their stability under learning in a substantial region of the parameter space. In this paper we have laid forth a theory that allows for the analysis of such equilibria in linear models with a predetermined variable. Both nonstochastic models and models with intrinsic random shocks were analyzed. We obtained existence for parameters in a proper subset of the region of indeterminacy. We have also shown existence of a related but distinct class of sunspot equilibria, namely those that are driven by finite-state Markov processes, but that take on infinitely many values even in nonstochastic models.

To analyze stability under learning we developed representations compatible with both classes of sunspot solutions, allowing us to establish the stability of sunspot equilibria for parameter values in a proper subset of the regions of existence. These theoretical results were supported by real-time simulations when agents learn using least squares estimators. The results of this paper indicate that stable finite-state Markov sunspot equilibria, and finite-state dependent sunspot equilibria, can arise quite generally, and stable noisy versions of these equilibria arise in models with intrinsic random shocks. Extension of these results to higher order and to multivariate models would be of considerable importance to applied macroeconomic models that incorporate both expectations and predetermined variables.

Appendix 1

Proof of Lemma 1. We employ the following notation: if A is an $n_1 \times m_1$ matrix and B is $n_2 \times m_2$ then their direct sum is an $(n_1 + n_2) \times (m_1 + m_2)$ matrix given by

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Let $\pi(i)$ be the matrix $(\pi_{nm}(i))$ and observe that if λ is a $k \times 1$ vector whose entries are each one, then

$$E_t s_{t+1} = \left(\oplus_i s'_{t-1}\pi(i)s_t\right)\lambda. \tag{26}$$

Then

$$\begin{aligned} E_t s'_t A s_{t+1} &= s'_t A \left(\oplus_i s'_{t-1}\pi(i)s_t\right)\lambda \\ &= \lambda' \left(\oplus_i s'_{t-1}\pi(i)s_t\right) A' s_t \\ &= s'_{t-1}\left[\pi(1)s_t, \cdots, \pi(k)s_t\right] A' s_t. \end{aligned} \tag{27}$$

Now let C^n be the $k \times k$ matrix whose m^{th}-column is the n^{th}-column of $\pi(m)$ and B be the matrix whose n^{th}-column is the n^{th}-column of $C^n A'$. Then for all possible s_t and s_{t-1} we have that the right hand side of (27) is equal to $s'_{t-1}Bs_t$.[13] ■

Proof of Proposition 2 (and Proposition 2'): It suffices to prove that if $\zeta_t = \overline{\zeta}_i \Leftrightarrow s_t = S_i$ is a solution to the homogeneous model then

$$\zeta_t = a\zeta_{t-1} + s'_{t-1}As_t \tag{28}$$

for some fixed point A of T_2, where a is a fixed point of T_1. Set $A_{ij} = \overline{\zeta}_j - a\overline{\zeta}_i$. Explicit computation shows $\zeta_t = \overline{\zeta}_i \Leftrightarrow s_t = S_i$ satisfies (28), so it remains to show A_{ij} is a fixed point of T_2. This requires an explicit formula for the T-map. The verbal description of the matrix $B(A)$ given in Lemma 3 yields the following form for the ij^{th} component of $T_2(A)$:

$$T_2(A)_{ij} = \beta(aA_{ij} + \sum_{m=1}^{k} \pi_{ij}(m)A_{jm}). \tag{29}$$

[13] For an explicit expression for $B(A)$ see the proof of Proposition 2.

Thus A is a fixed point if and only if

$$(\frac{1}{\beta} - a)A_{ij} = \sum_{m=1}^{k} \pi_{ij}(m)A_{jm}. \tag{30}$$

This set of k^2 linear homogeneous equations always has zero as a solutions; nonzero solutions exist only in case of linear dependency, which will not hold in general. However, because $\overline{\zeta}_t$ is a sunspot equilibrium, we have that equations (3) hold, that is

$$\overline{\zeta}_j - \delta\overline{\zeta}_i = \beta \sum_{m=1}^{k} \pi_{ij}(m)\overline{\zeta}_m. \tag{31}$$

We proceed to show that if $A_{ij} = \overline{\zeta}_j - a\overline{\zeta}_i$ then (31) implies (30). We have (31)

$$\begin{aligned}
&\Leftrightarrow \quad \delta(\overline{\zeta}_j - a\overline{\zeta}_i) = (\delta - a)\overline{\zeta}_j + \beta a \sum \pi_{ij}(m)\overline{\zeta}_m \\
&\Leftrightarrow \quad \delta(\overline{\zeta}_j - a\overline{\zeta}_i) = -\beta a^2\overline{\zeta}_j + \beta a \sum \pi_{ij}(m)\overline{\zeta}_m \\
&\Leftrightarrow \quad \delta(\overline{\zeta}_j - a\overline{\zeta}_i) = \beta a \sum \pi_{ij}(m)(\overline{\zeta}_m - a\overline{\zeta}_j) \\
&\Leftrightarrow \quad \delta A_{ij} = \beta a \sum \pi_{ij}(m)A_{jm} \\
&\Leftrightarrow \quad (a - a^2\beta)A_{ij} = \beta a \sum \pi_{ij}(m)A_{jm},
\end{aligned}$$

and this last line holds if and only if (30) holds.■

Proof of Proposition 4. The proof of this Proposition uses the following Lemma.

Lemma 2: *Let π satisfy the existence restrictions (13). If $T_2(a, A) = A$ then*

$$A_{ij} = \frac{A_{jj}}{1-a} - \frac{aA_{ii}}{1-a}.$$

Proof of Lemma 2: Let $A_{12} = 1$. Since A is a fixed point, we have that $A_{ij} = K_{ij}A_{12}$. Thus it suffices to show $K_{ij} = \frac{K_{jj}}{1-a} - \frac{aK_{ii}}{1-a}$. This can be shown using Maple.■

Proof of Proposition 4. Here we state and prove the generalization of this proposition to the stochastic model (19):

Proposition 4' *Suppose β and δ are such that 2-SSEs exist and the roots of the associated quadratic are real. Suppose a denotes a root of this quadratic and $b = (1-\beta a)^{-1}$. Let π satisfy the existence restrictions (13). If $T_2(a, A) = A$ and*

$$y_t = ay_{t-1} + s'_{t-1}As_t + v_t,$$

then y_t is a noisy 2-SSE.

For the nonstochastic case simply set $bv_t = 0$ in the following proof.

Proof of Proposition 4': For each realization of the process s_t there is a map $\tau : \mathbb{Z} \to \{1, 2\}$ such that $s_t = S_i \Leftrightarrow \tau(t) = i$. Thus we can write (4) as

$$(1 - aL)y_t = A_{\tau(t-1)\tau(t)} + bv_t, \tag{32}$$

where L is the lag operator, i.e. $Ly_t = y_{t-1}$. Now let $\hat{A} = vec(A)$ and for reasons to be clear later, index $\hat{A}$ starting with 0, that is, $\hat{A} = (\hat{A}_0, \hat{A}_1, \hat{A}_2, \hat{A}_3)'$. Let $\sigma : \mathbb{Z} \to \{0, 1, 2, 3\}$ be defined by

$$\sigma(t) = \hat{\tau}(t-1) + \hat{\tau}(t-1)\hat{\tau}(t) + \hat{\tau}(t) + \hat{\tau}(t-1)(1 - \hat{\tau}(t)),$$

where $\hat{\tau} = \tau - 1$. Then $\hat{A}_{\sigma(t)} = A_{\tau(t)\tau(t-1)}$. This allows us to write equation (32) as

$$(1 - aL)y_t = \hat{A}_{\sigma(t)} + bv_t. \tag{33}$$

The Lemma implies

$$\hat{A}_{\sigma(t)} = A_{\tau(t-1)\tau(t)} = \frac{1}{1-a}(1 - aL)A_{\tau(t)\tau(t)}.$$

Combine this with (33), and using stationarity, which allows us to cancel the lag polynomial $(1 - aL)$, yields

$$y_t = \frac{1}{1-a}A_{\tau(t)\tau(t)} + \frac{b}{1-aL}v_t,$$

which is a 2-SSE. ■

Proof of Proposition 5. Label the real roots as a_i and notice that $1/\beta - a_i = a_j \equiv \alpha$ for $i, j = 1, 2$ with $i \neq j$. Thus we can write the conditions for a fixed point of the T-map as

$$\alpha A_{11} = \pi_{11}A_{11} + (1 - \pi_{11})A_{12} \tag{34}$$
$$\alpha A_{12} = \pi_{12}A_{21} + (1 - \pi_{12})A_{22} \tag{35}$$
$$\alpha A_{21} = \pi_{21}A_{11} + (1 - \pi_{21})A_{12} \tag{36}$$
$$\alpha A_{22} = \pi_{22}A_{21} + (1 - \pi_{22})A_{22} \tag{37}$$

where $|\alpha| < 1$ by indeterminacy. We proceed as follows: fix α and show that we can choose A_{ij} and π_{ij} so that the above conditions are satisfied. The key observations are that, for each equation, the left hand side is a convex combination of the A_{ij} on the right hand side, and that the π_{ij} are independent across equations.
Case 1: $\alpha \geq 0$. Choose A_{ij} so that

$$A_{22} < A_{12} < 0 < A_{21} < A_{11}. \tag{38}$$

Case 2: $\alpha < 0$. Choose A_{ij} so that

$$-A_{12} < A_{21} < \alpha A_{12} < A_{11} < 0 < A_{22} < -A_{21} < A_{12}. \tag{39}$$

Case 1 implies

$$\begin{aligned}
\alpha A_{11} &\in (A_{12}, A_{11}) \\
\alpha A_{12} &\in (A_{22}, A_{21}) \\
\alpha A_{21} &\in (A_{12}, A_{11}) \\
\alpha A_{22} &\in (A_{22}, A_{21}),
\end{aligned}$$

and Case 2 implies similar set membership except that the endpoints of each of the intervals are reversed. Equations (34) - (37) follow immediately from the implied choice of π. It remains to show that if $(\beta, \delta) \in A_1$ then there exist π not satisfying (13) such that $\Omega_1(\pi)$ is not trivial. Given the choices of A_{ij}, the associated transition array can be constructed as follows:

$$\begin{aligned}
\pi_{11} &= (A_{11} - A_{12})^{-1}(\alpha A_{11} - A_{12}) \\
\pi_{12} &= (A_{21} - A_{22})^{-1}(\alpha A_{12} - A_{22}) \\
\pi_{21} &= (A_{11} - A_{12})^{-1}(\alpha A_{21} - A_{12}) \\
\pi_{22} &= (A_{21} - A_{22})^{-1}(\alpha A_{22} - A_{22})
\end{aligned}$$

Begin by noticing that, according to (13), the set of π to which correspond 2-SSEs is one dimensional, being pinned down by the choice of π_{22}. Now notice the choice of A_{21} and A_{22} determines π_{22}. On the other hand, in both case 1 (case 2), for given choice of A_{21} and A_{22} there are multiple A_{12} and A_{11} which satisfy the restriction (38) ((39)). In particular, the choice of A_{21} and A_{22} does not pin down the values of π_{11}, π_{12} and π_{21}. This shows the set of π to which correspond non-trivial $\Omega_1(\pi)$ has dimension greater than one, thus completing the proof. ■

Proof of Proposition 6. Clearly R, the set of π satisfying (13), is a line segment and is therefore homeomorphic to $\mathbb{R}$. Let $\Omega(\pi)$ be the set of fixed points of the T-map. If $(a, A, b) \in \Omega(\pi)$ and $A \neq 0$, then the process

$$y_t = ay_{t-1} + s'_{t-1}As_t + bv_t$$

is a 2-state dependent sunspot equilibrium, and if the associated $\pi \notin R$ then this process is an ergodic 2-SSEs. Let $\hat{R}$ be the set of all $\pi \in I$ so that $\Omega(\pi)$ is non-trivial, that is, so that $\Omega(\pi)$ contains points other than $(a, 0, b)$. Every 2-SSE is a 2-state dependent sunspot equilibrium, thus $R \subset \hat{R}$. Let a be a fixed point of T_1, $\alpha = 1/\beta - a$, and define a map $M : I \to \mathbb{R}^{4\times4}$ by

$$M(\pi) = \begin{bmatrix} \alpha - \pi_{11} & 0 & \pi_{11} - 1 & 0 \\ -\pi_{21} & \alpha & \pi_{21} - 1 & 0 \\ 0 & -\pi_{12} & \alpha & \pi_{12} - 1 \\ 0 & -\pi_{22} & 0 & \alpha + \pi_{22} - 1 \end{bmatrix},$$

and set $\Gamma : I \to \mathbb{R}$ by $\Gamma = \det \circ M$. Note that A is a fixed point of T_2 if and only if $M \cdot vec(A) = 0$. This may be seen using equations (34)-(37) in the paper. Thus

$$\hat{R} = \{\pi \in I : \Gamma(\pi) = 0\}.$$

Now notice that Γ is a polynomial in π_{ij}. The gradient of this polynomial can be explicitly computed and shown not to vanish on $\hat{R}$. The implicit function theorem then applies to show that about any point at which the gradient does not vanish there is a neighborhood homeomorphic to $\mathbb{R}^3$. That $\hat{R}$ has measure zero in I follows from the fact that it is the zero-set of a non-zero polynomial. ■

Proof of Proposition 7. Assume $\pi \in \hat{R} \setminus R$. It is straightforward to show that all non-trivial fixed points of the T-map have the property that there are at least two distinct values among the A_{ij}, and we write

$$y_t = \sum_{n=0}^{\infty} a^n \hat{A}_{\sigma(t-n)}.$$

We would like to know about the possible values obtained by y_t; denote this set by Y. We proceed as follows: fix t and consider realizations of the process s_t up to time t. Note that each realization identifies a sequence $\{\hat{A}_n\}_{n=-\infty}^{t}$, and any such sequence is possible. It is immediate that Y contains infinitely many points. For example, set

$$z_k = \sum_{n \neq k}^{\infty} a^n \hat{A}_1 + a^k \hat{A}_2,$$

where we are assuming $\hat{A}_1 \neq \hat{A}_2$. Then $z_k \in Y$, and $i \neq j \Rightarrow z_i \neq z_j$. It is somewhat more difficult to show that y_t takes on infinity many values with positive probability. We now turn to this problem.

Let I be the unit interval in $\mathbb{R}$ and express all elements of I in base 4. Specifically, for $\gamma \in I$, write

$$\gamma = \sum_{k=1}^{\infty} \gamma_k \left(\frac{1}{4}\right)^k,$$

where $\gamma_k \in \{0, 1, 2, 3\}$. Define $f : I \to \mathbb{R}$ by

$$f(\gamma) = \sum_{n=1}^{\infty} a^{n-1} \hat{A}_{\gamma_n}.$$

Note that $Y = f(I)$.

Lemma 3 *The function f is continuous.*

Proof: Let $\varepsilon > 0$ and $\varepsilon' = (1/4)^n$. Let $A^s = max_i\{|\hat{A}_i|\}$. Notice that if $|\gamma - \gamma'| < \varepsilon'$ then $\gamma_k = \gamma'_k$ for $k < n$. Thus

$$|f(\gamma) - f(\gamma')| \leq 2 \sum_{m=n+1}^{\infty} |a|^m A^s.$$

Note that rearrangement of the sums is legitimate because the series are absolutely convergent. The right hand side goes to zero as $n \to \infty$ and so can be made smaller than ε. ■

We know that continuous functions send connected sets to connected sets, and that connected subsets of $\mathbb{R}$ are intervals. Noting that $f(i/4) \neq f(j/4)$ if $\hat{A}_i \neq \hat{A}_j$ shows that Y contains an interval. It follows that there are uncountably many possible values of y_t.

The next step is to note that the Markov process s_t induces a probability measure μ on the interval I. Specifically, each realization $\hat{s}$ of the process

$\{s_t\}$ induces a function $\sigma(\hat{s})$ which may, in turn be thought of as a realization of a four state first order Markov process.[14] To each σ is identified a real number $\gamma \in I$ whose base four expansion has $\sigma(t-k+1)$ as the k^{th} coefficient γ_k. Now let E be any Borel set in I. Then $\mu(E) = prob(\sigma(\hat{s}) \in E)$, or, in words, the measure of the set E is the probability that the realization $\sigma(\hat{s})$ is identified with some element of E. Notice that, by construction, if $E \subset Y$ then $prob(y_t \in E) = \mu(f^{-1}(E))$ where

$$f^{-1}(E) = \{x \in I | f(x) \in E\}$$

denotes the pre-image of E under f.

Lemma 4 *If $U \subset I$ is open then $\mu(U) > 0$.*

Proof: Let U be open. Then U contain an interval (α, β). Within this interval it is straightforward to construct an interval $J = (\hat{\alpha}, \hat{\beta})$ so that there exists n with $\hat{\alpha}_m = 0$ and $\hat{\beta}_m = 0$ for $m > n$, where $\hat{\alpha}_m$ is the m^{th} component of the base four expansion of α. We claim that $\mu(J) > 0$. To see this, begin with the simple case that $m = 1$ and $J = (1/4, 3/4)$. Then $\mu(J)$ is the unconditional probability that $\sigma(t) \in \{1, 2, 3\}$. Since all transition arrays have full support, this probability is non-zero. Now consider the general case. Set

$$P = \{(\delta_1, \cdots, \delta_n) | \delta_i \in \{\hat{\alpha}_i, \cdots, \hat{\beta}_i\}\}.$$

Now notice that $\mu(J)$ is the unconditional probability that

$$(\sigma(t), \cdots, \sigma(t-n)) \in P.$$

This probability is non-zero because the transition arrays have full support. ■

We can now finish the proof of Proposition **7** by showing that with positive probability y_t takes infinitely many values. Suppose not. Then there exists $F \subset Y$ such that F is finite and $\mu(f^{-1}(F)) = 1$. Recall there is an interval I_Y in Y. Since F is closed, $O = (\mathbb{R} \setminus F) \cap I_Y$ is open and non-empty. Let $U = f^{-1}(O) \subset I$. By Lemma 1, U is open and by Lemma 2, $\mu(U) > 0$. But $U \subset f^{-1}(\mathbb{R} \setminus F)$ and $\mu(f^{-1}(\mathbb{R} \setminus F)) = 0$. Thus we reach the required contradiction. ■

[14] Note that all relevant transition arrays have full support. Thus, given any state, each state is reachable with positive probability.

Proof of Proposition 9: To prove stability we must show that when

$$\beta < 0 \ \ \delta > 1 + \beta \ \ \delta < -1 - \beta, \ \ \beta\delta < 1/4,$$

it follows that the eigenvalues of $\frac{\partial vec(T_2)}{\partial vec(A)}$, given in (18), have real part less than or equal to one. Imposing the restrictions on π given in equation (13), substituting $\beta = \frac{a_1 - \delta}{a_1^2}$ into (18) and using Mathematica to compute the eigenvalues we obtain

$$1, \ \ 2\left(1 - \frac{\delta}{a}\right), \ \ 1 - \frac{\delta(1+a)}{a}, \ \ (1+a)\beta,$$

where for notational simplicity we write $a = a_1$. Substituting $\delta = a(1 - \beta a)$ it follows that the non-trivial eigenvalues are

$$2\beta a, \ \ a(\beta(1+a) - 1), \ \ \beta(1+a).$$

Clearly $2\beta a = 1 - \sqrt{1 - 4\beta\delta} < 1$ since $1 - 4\beta\delta > 0$. Next,

$$a(\beta(1+a) - 1) < 1 \Leftrightarrow \beta a(1+a) < 1 + a \Leftrightarrow \beta a < 1,$$

where the final implication follows from the fact that $a \in (0, 1)$. Also, we know $\beta a < 1$ since $2\beta a$. Finally, $\beta < 0$ and $1 + a > 0$ implies $\beta(1+a) < 1$. This establishes stability of 2-SSEs in region A_1.

Suppose instead that β and δ are in Area B_1. To see that in this region an associated 2-SSE is unstable under learning, note that E-stability would require

$$\begin{aligned} \beta(1+a) \leq 1 \quad &\Leftrightarrow \quad 2\beta a < 2 - 2\beta \\ &\Leftrightarrow \quad 2\beta - 1 \leq \sqrt{1 - 4\beta\delta} \\ &\Leftrightarrow \quad \beta - 1 \leq -\delta. \end{aligned}$$

But in region B_1, $\delta > 1 - \beta$, which provides the required contradiction.■

Appendix 2

In this section we prove the result described in Section 4 and used in Section 5.4 that if the derivative of $T(\theta) - \theta$ has a single root of zero then the necessary condition for E-stability are also sufficient.

Consider the following system of differential equations:

$$\begin{aligned} \frac{da}{d\tau} &= g(a) \\ \frac{d\theta}{d\tau} &= B(a(\tau))\theta \end{aligned}$$

with $a \in \mathbb{R}$ and B an $n \times n$ matrix with entries exhibiting linear dependence on the value $a(\tau)$. Assume (5) has $\bar{a}$ as a locally asymptotically stable fixed point. A result from (Hirsch and Smale 1974), page 181m shows that there is a neighborhood U of $\bar{a}$ so that $a \to \bar{a}$ exponentially, that is, there exists a constant K and $\alpha < 0$ so that if $a(0) \in U$ then

$$|a(\tau) - \bar{a}| < Ke^{\alpha\tau}|a(0) - \bar{a}|.$$

We will prove the following result:

Proposition 11 *If $B(\bar{a})$ has one zero eigenvalue and each of the remaining eigenvalues have negative real part then, locally, $\theta(\tau)$ converges to a finite value.*

To prove this proposition, we need the following lemma, which is just a direct application of the implicit function theorem.

Lemma 5 *Let $P : R \times R^{n+1} \to R$ be a collection of degree n polynomials in x with coefficients b:*

$$P(x, b) = \sum_{k=0}^{n} b_k x^k.$$

Let $f : R \to R^{n+1}$ be C^k in each entry. Define $F : R^2 \to R$ by $F(x, y) = P(x, f(y))$. If $\frac{\partial F}{\partial x} \neq 0$, then, locally, the equation $F(x, y) = 0$ defines x as an implicit function of y. This function is C^k.

The requirement that $\frac{\partial F}{\partial x} \neq 0$ says that the root x of the polynomial determined by the coefficients $f(y)$ is not repeated.

Proof of Proposition 11: Assume $a(0) \in U$. Write the real Jordan decomposition of $B(a)$ as $Q(a)B(a)Q(a)^{-1} = \Lambda(a)$ with Λ block diagonal, and assume the eigenvalues are ordered so that $\Lambda_{11}(\bar{a}) = 0$. Change coordinates to $z = Q(a)^{-1}\theta$. Because $\Lambda_{11}(a(\tau)) \to 0$ and because there is a unique zero eigenvalue it follows that locally $\Lambda_{11}(a)$ is real (if it were complex, its conjugate would also converge to zero). Therefore, the differential system (6) becomes

$$\frac{dz}{d\tau} = \left(\Lambda_{11}(a(\tau)) \oplus \hat{\Lambda}\right) z,$$

and we may isolate the differential equation

$$\frac{dz_1}{d\tau} = \Lambda_{11}(a(\tau))z_1.$$

This differential equation is separable, and so may simply be integrated. We obtain:

$$z_1(\tau) = z_1(0)e^{\int_0^\tau \Lambda_{11}(a(u))du}.$$

Since the remaining eigenvalues of $B(a)$ have real part less than zero, our result is proved by showing

$$\lim_{\tau \to \infty} \int_0^\tau \Lambda_{11}(a(u))du < \infty.$$

To this end, note that since $\Lambda_{11}(a)$ is (locally) an eigenvalue of unit multiplicity, it is therefore a non-repeated root of the characteristic polynomial of the matrix $B(a)$. The above Lemma then says that Λ_{11} is C^k in a, for k larger than one (since the entries of B are linear in a). So we may apply Taylor's theorem to obtain that

$$\Lambda_{11}(a) = \sum_{n=1}^{k-1} \frac{\Lambda_{11}^{(n)}(\bar{a})}{n!}(a - \bar{a})^n + \frac{\Lambda_{11}^{(k)}(\xi)}{k!}(a - \bar{a})^k$$

for some ξ near $\bar{a}$. Therefore

$$\begin{aligned}|\Lambda_{11}(a(\tau))| &\leq K\sum_{n=1}^{k-1}\frac{\left|\Lambda_{11}^{(n)}(\bar{a})\right|}{n!}e^{\alpha n\tau}|a(0)-\bar{a}|^n + \frac{\left|\Lambda_{11}^{(k)}(\xi)\right|}{k!}|a(0)-\bar{a}|^k e^{\alpha k\tau} \\ &\leq \left(K\sum_{n=1}^{k-1}\frac{\left|\Lambda_{11}^{(n)}(\bar{a})\right|}{n!}|a(0)-\bar{a}|^n + \frac{\left|\Lambda_{11}^{(k)}(\xi)\right|}{k!}|a(0)-\bar{a}|^k\right)e^{\alpha\tau}.\end{aligned}$$

Since Λ_1 is C^k, it follows that locally the error term in the Taylor expansion is uniformly bounded. We conclude that there is a constant $\hat{K}$ so that $|\Lambda_{11}a(\tau))| \leq \hat{K}e^{\alpha\tau}$. Then

$$\int_0^\tau |\Lambda_{11}(a(u))|\,du \leq \hat{K}\int_0^\tau e^{\alpha u}du,$$

and the right hand side converges to a finite limit as $\tau \to \infty$. ■

To apply this proposition to obtain the result claimed, we simply note that $\mathrm{vec}(T_2(a, A)) = B\mathrm{vec}(A)$ with the coefficients of the matrix B linear in the variable a.

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Figure 1

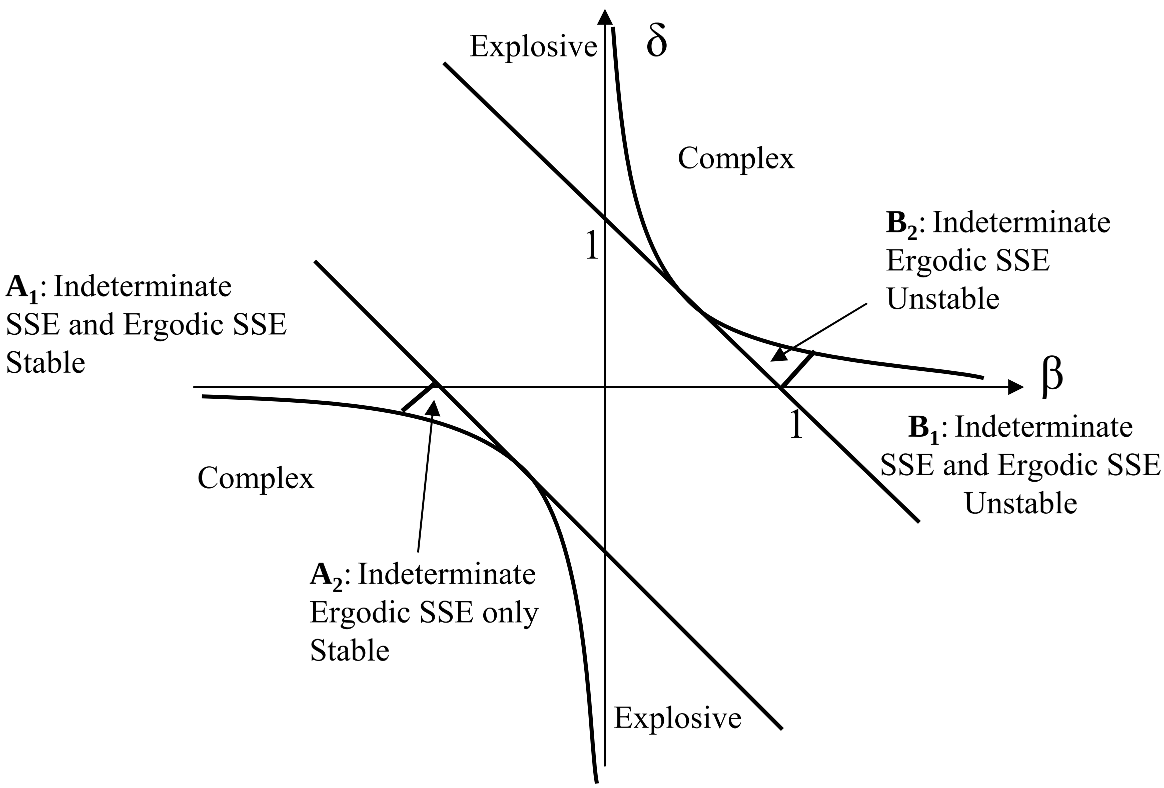

Explosive
δ
Complex
1
B₂: Indeterminate
Ergodic SSE
Unstable
A₁: Indeterminate
SSE and Ergodic SSE
Stable
β
1
B₁: Indeterminate
SSE and Ergodic SSE
Unstable
Complex
A₂: Indeterminate
Ergodic SSE only
Stable
Explosive

Figure 2.1: Ergodic 2-SSE, β = -3, δ = 1

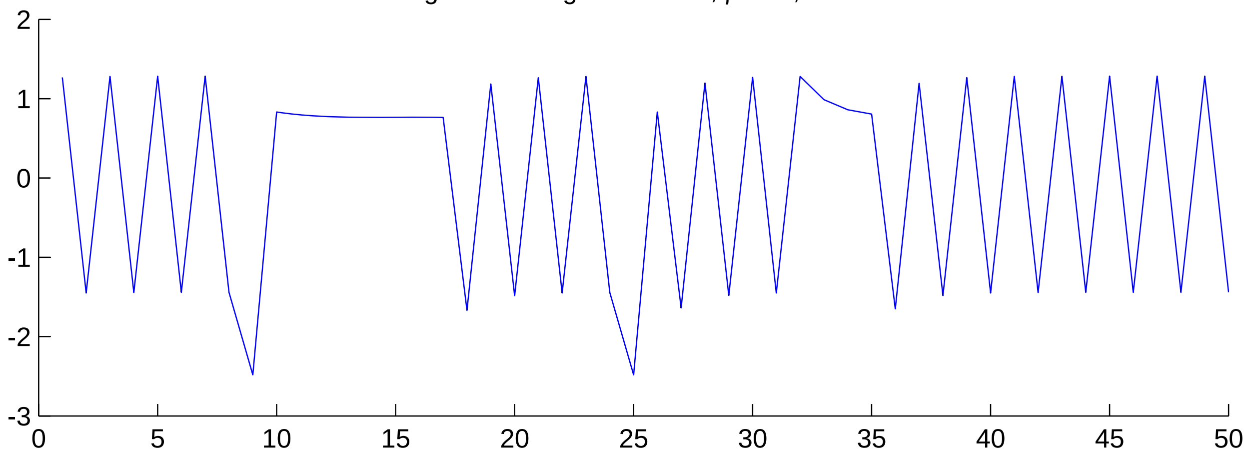
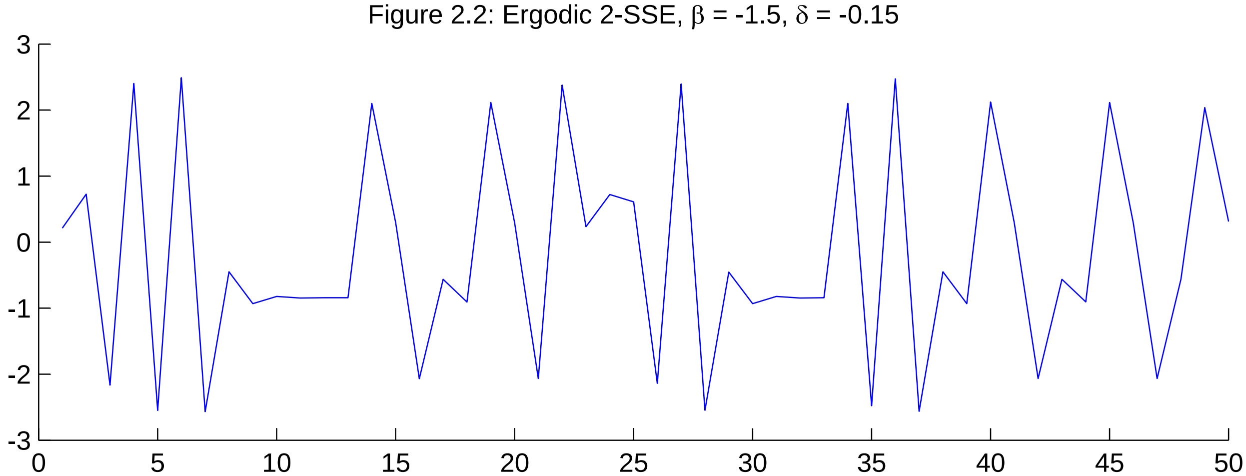


Figure 2.2: Ergodic 2-SSE, β = -1.5, δ = -0.15



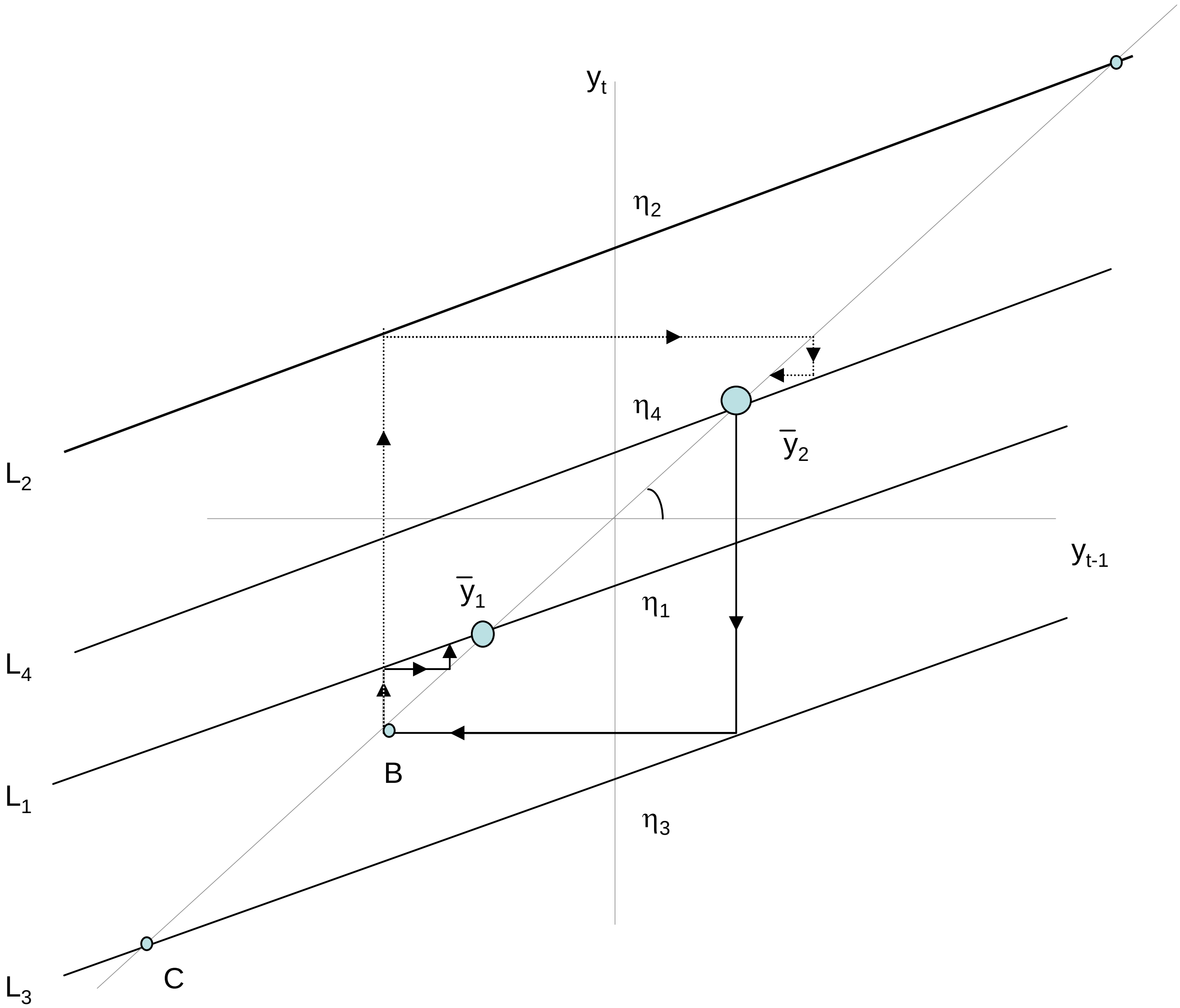
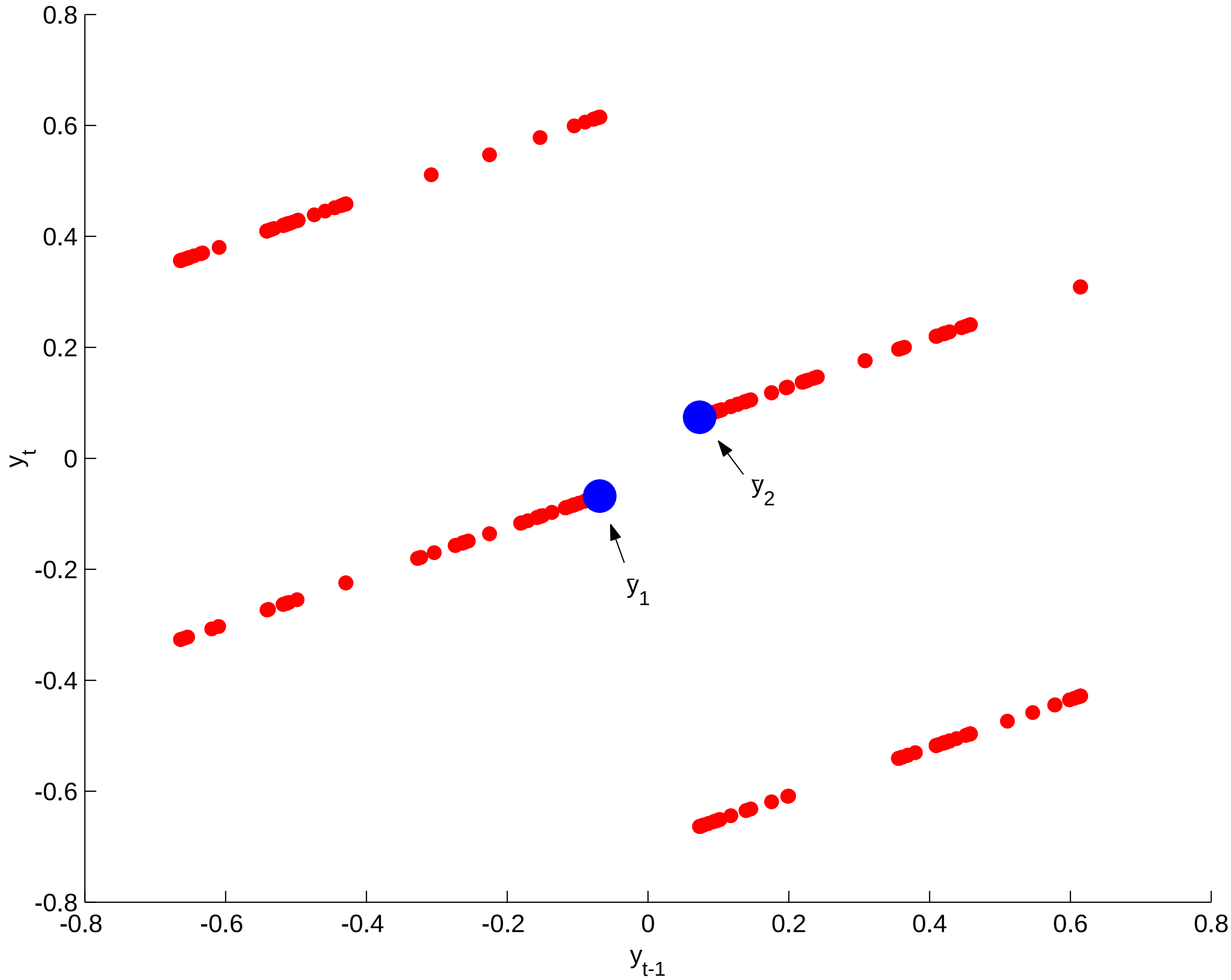


Figure 3: Ergodic 2-SSE

Figure 4: β = -3, δ = 1, π_{11} = 0.9, π_{12} = 0.16, π_{21} = 0.73, π_{22} = 0.1



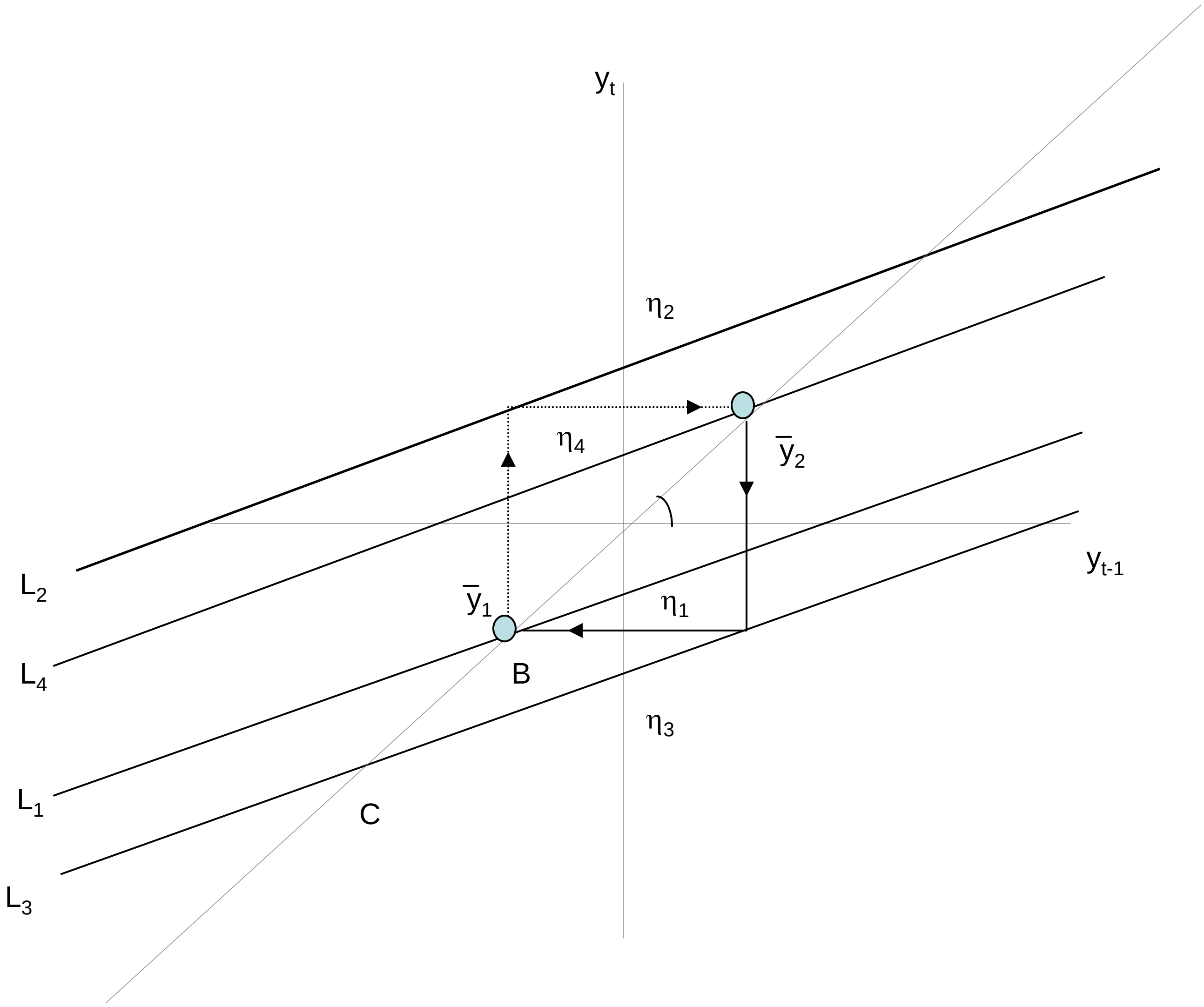


Figure 5: 2-SSE

Figure 6: Learning a 2 SSE